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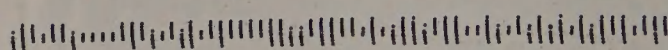
MAY 2014



technology-
enhanced
discovery

THE SIERPINSKI TRIANGLE
multiple representations

TOWERS OF HANOI
and binary notation



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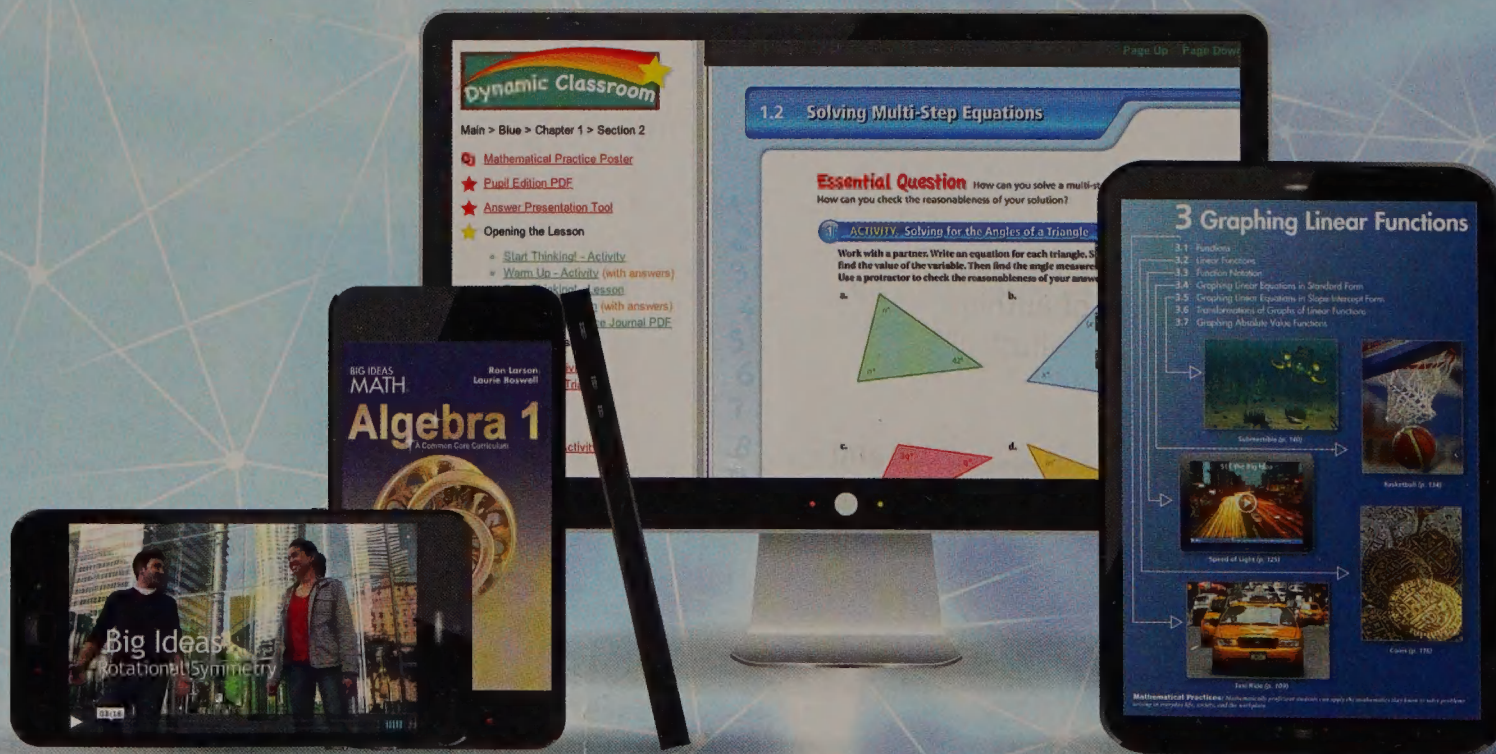
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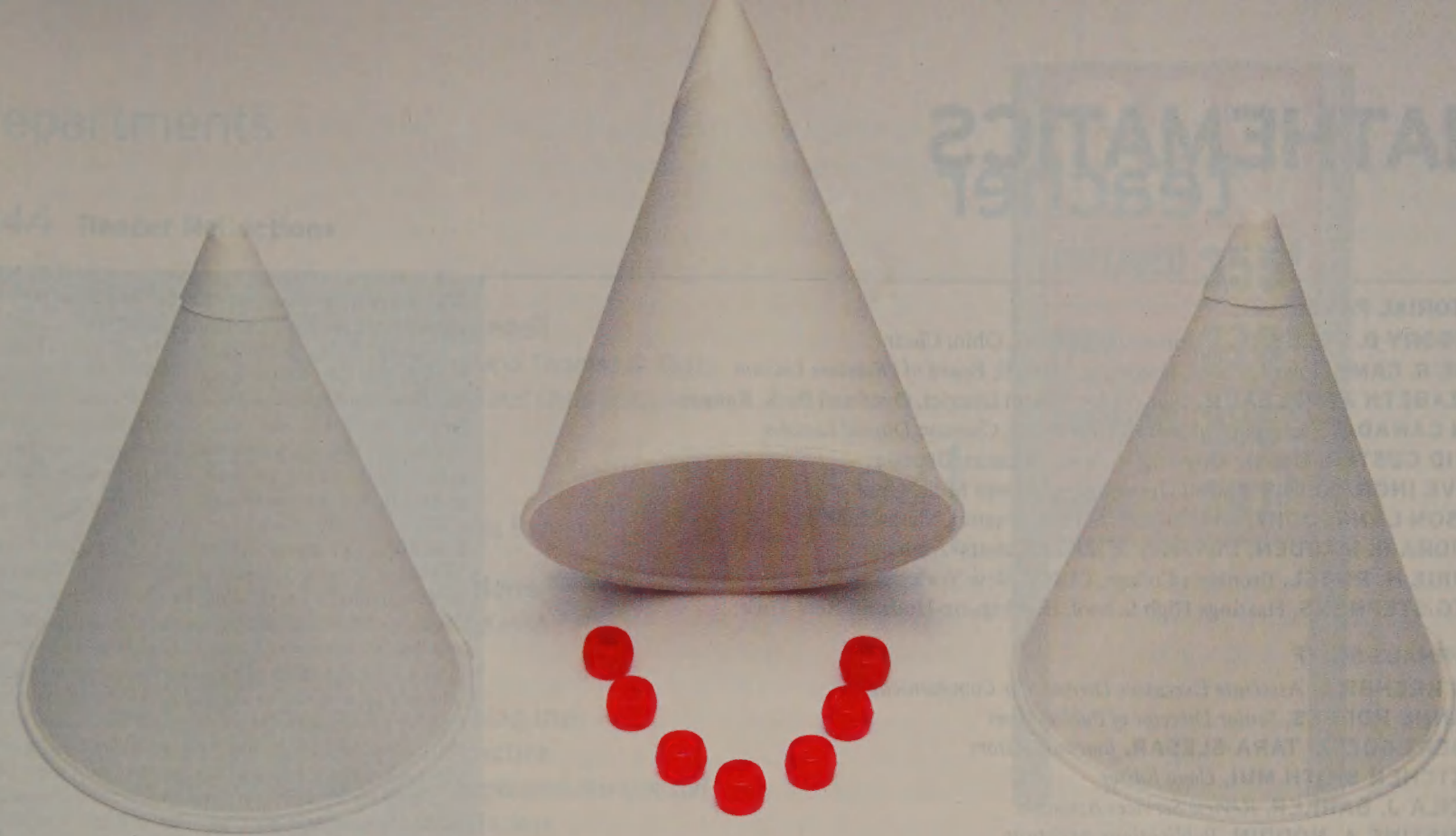
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contents

Volume 107, Number 9, May 2014

features

- 656 Principles to Actions: Mathematics Programs as the Core for Student Learning**
Daniel Brahier, Steve Leinwand, and DeAnn Huinker
- 660 Technology-Enhanced Discovery**
Chris Harrow and Lillian Chin
Tracking the foci of conic sections reveals a surprising locus.
- 666 Multiple Representations and Connections with the Sierpinski Triangle**
J. Vince Kirwan and Jennifer M. Tobias
A task using multiple representations helps students write explicit algebraic equations.
- 672 Authenticity of Mathematical Modeling**
Dung Tran and Barbara J. Dougherty
The choice and context of authentic problems—such as designing a staircase or a soda can—illustrate the modeling process in several stages.
- 688 Equivalent Expressions Using CAS and Paper-and-Pencil Techniques**
Nicole L. Fonger
Designing activities to reconcile multiple representations supports students' focus and fluency.

on the cover

Hidden mathematical relationships are uncovered when familiar objects are seen from a new perspective. Using available graphing technology, teacher Chris Harrow tracks properties of conic sections and passes the focus to student Lillian Chin, whose investigation leads her to research and proof. They share their discovery and convey their enthusiasm in an enhanced learning experience (see pages 660–65).

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departments

644 Reader Reflections

648 Media Clips

Mosquitoes // Pythagorean Baseball

John Donovan // Gregory D. Foley and Thomas R. Butts

Edited by Louis Lim, Lionel Garrison, and Chris Bolognese

652 Mathematical Lens

Seeing the Light

Edited by Ron Lancaster and Brigitte Bentele

680 May 2014 Calendar and Solutions

Edited by Margaret Coffey and Art Kalish

694 Activities for Students

Predicting Future Gas Prices Using the Standards for Mathematical Practice

Stephen F. Bismarck, Jeremy Zelkowsky, and Jim Gleason

Edited by Ruth Dover and Patrick Harless

701 Connecting Research to Teaching

Evaluating and Writing Dynamic Geometry Tasks

Aaron Trocki

Edited by Margaret Kinzel and Laurie Cavey

706 Technology Tips

High-Leverage iPad Apps for the Mathematics Classroom

Ayanna D. Perry, Emily P. Thrasher, and Hollylynne S. Lee

Edited by Larry Ottman and James Kett

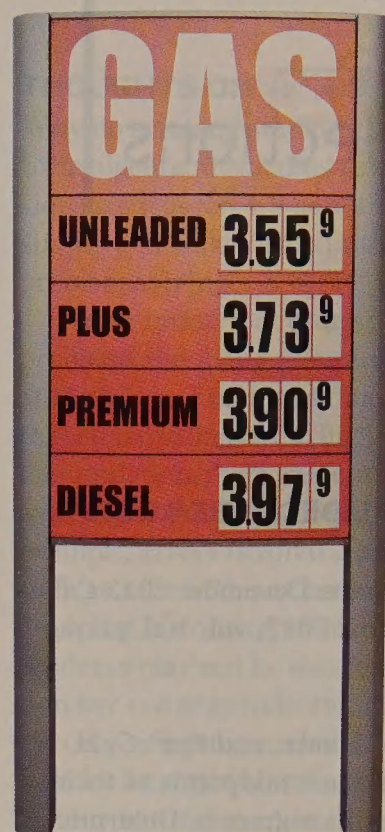
712 Delving Deeper

Twists on the Tower of Hanoi

James Metz

Edited by Maurice Burke, J. Kevin Colligan, Maria Fung, and Jeffrey J. Wanko

694



716 For Your Information Publications

718 Thanks from the Editorial Panel

720 The Back Page: My Favorite Lesson

Algebra about Me

Victoria Miles

Edited by Roger Day

calls for manuscripts

687 2016 Focus Issue: Teaching Mathematics Online

705 Connecting Research to Teaching

more4U

- Reader Reflections (p. 644): A proof of a Pythagorean triple generator (PDF)
- Activities for Students (p. 694): Customizable activity sheets (.doc)
- The Back Page (p. 720): Photographs (PDF) and an activity sheet (.doc)



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in the next issue

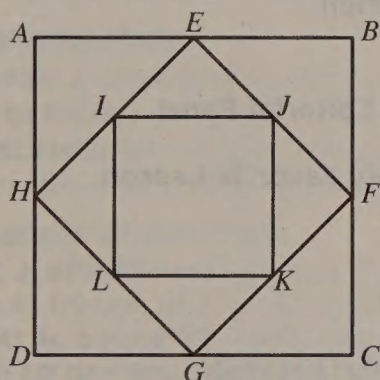
Coming in August 2014:

- "Who's in the Lead?" by Alfinio Flores
- "Triangles from Three Points," by Wayne Nirode
- "Addressing the Mathematical Practices in Calculus," by Mary Pilgrim

PROBLEM 11, DECEMBER 2012 CALENDAR

Problem 11 in the December 2012 Calendar (MT Nov. 2012, vol. 106, no. 4, p. 280) states:

$ABCD$ is a square, and $E, F, G, H, I, J, K,$ and L are midpoints of their respective line segments. Determine the area of $IJKL$ if the area of $ABCD$ is 12.



Rather than restrict ourselves to a square—or, for that matter, the proportion one-half—herein we connect sequential points that are proportionately p down the sides of the general triangle and the general quadrilateral. The problem is to find the ratio of the area of the inscribed polygon to the area of the original. Our tool is the knowledge that the area of a triangle is $0.5st \sin \alpha$, where α is the angle between sides s and t .

We appreciate the interest and value the views of those who write. Readers commenting on articles are encouraged to send copies of their correspondence to the authors. For publication: All letters for publication are acknowledged. Letters to be considered for publication should be in MS Word document format and sent to mt@nctm.org. Letters should not exceed 250 words and are subject to abridgment. At the end of the letter include your name and affiliation, if any, including e-mail address, per the style of the section.

We begin with the triangle (see **fig. 1** [Siegel]). Each side is segmented proportionately into p and $1 - p$ of its full length. When we connect the corresponding proportional points with line segments, we inscribe a triangle, forming three triangles between the original triangle and the inscription. We may choose to express the area of the original triangle as $(0.5ab \sin C + 0.5ac \sin B + 0.5bc \sin A)/3$. This is one-third of three different expressions of the area. The sum of the areas of the three aforementioned triangles is $0.5p(1 - p)ab \sin C + 0.5p(1 - p)ac \sin B + 0.5p(1 - p)bc \sin A$. The ratio of the summed area to the area of the original triangle is $3p(1 - p)$. Therefore, the ratio of the areas of the

inscribed and original triangles is $1 - 3p(1 - p)$, which we may write as $3p^2 - 3p + 1$. If we use $p = 1/n$, then the ratio becomes $(n^2 - 3n + 3)/n^2$.

Now we consider the analogous situation for the quadrilateral (see **fig. 2** [Siegel]). We may choose to express the area of the quadrilateral as $([0.5ab \sin A + 0.5cd \sin C] + [0.5bc \sin B + 0.5ad \sin D])/2$. This is half the sum of two different expressions of that area. The total area of the four triangles bounded by the two quadrilaterals is $0.5p(1 - p)ab \sin A + 0.5p(1 - p)cd \sin C + 0.5p(1 - p)bc \sin B + 0.5p(1 - p)ad \sin D$. The ratio of this summed area to that of the original quadrilateral is $2p(1 - p)$. Thus, the ratio of the areas of the

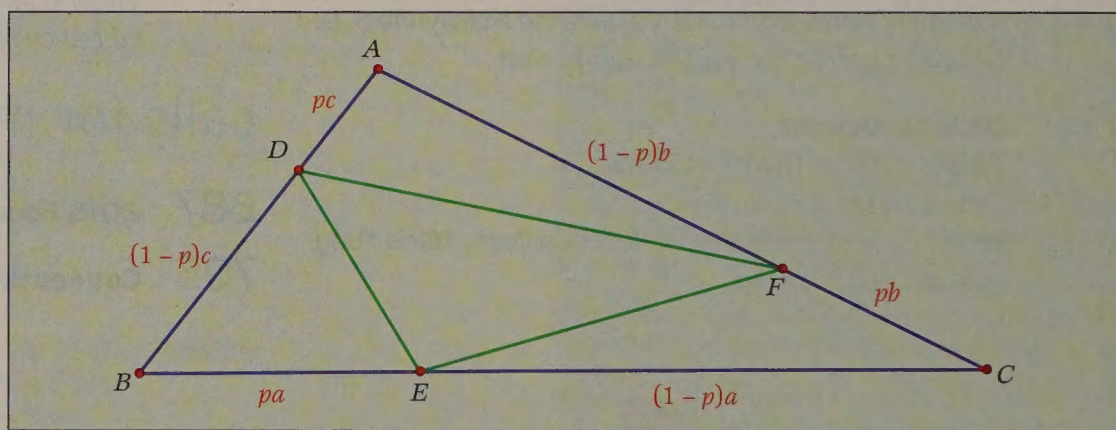


Fig. 1 (Siegel)

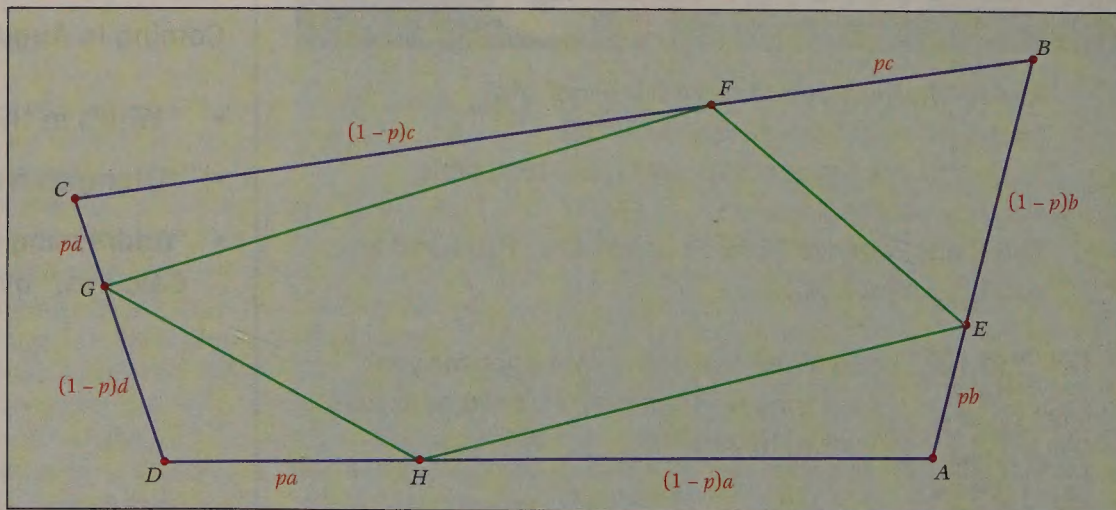


Fig. 2 (Siegel)

two quadrilaterals is $2p^2 - 2p + 1$. If we use $p = 1/n$, then the ratio becomes $(n^2 - 2n + 2)/n^2$.

It is remarkable that, for any nonspecific triangle or quadrilateral, the ratio of the inscribed figure's area to the original area is a function of only the proportion, not the sides. This is not true for other general polygons. We should also marvel that the expressions of those ratios are such simple quadratics in p .

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Niagara Falls, NY, July 8, 2013

PROBLEM 12, AUGUST 2013 CALENDAR

I have an answer different from that published for problem 12 in the August 2013 Calendar. The original problem (MT Aug. 2013, vol. 107, no. 1, p. 40) states:

A school bus has 15 rows with two benches each; each bench seats three. Sara prefers to sit directly next to Megan. Tomas prefers to sit next to Sara, possibly across the aisle from her. Megan has no preferences. No child sits in the same seat twice. If the children ride the bus to school and back every day, how many days will pass before they break one of their rules?

If we allow the following arrangements of students using the same notation as published in the presented solution, we could have four arrangements for each row that do not break the established rules: Tomas sits in seats 1, 4, 3, and 6 in each row; Sara sits in seats 2, 3, 4, and 5 in each row; and Megan sits in seats 3, 2, 5, and 4 in each row (see **fig. 1 [Beeken]**).

This arrangement allows thirty days of unique seating before one of the rules

1	2	3	4	5	6
T	S	M	X	X	X
X	M	S	T	X	X
X	X	T	S	M	X
X	X	X	M	S	T

Fig. 1 (Beeken)

must be broken. Sara must never sit by a window because Sara cannot sit by Megan while Tomas sits next to Sara if Sara chooses a window seat. Whenever Sara takes a middle seat, Tomas sits near a window, so he can still use the aisle seat for the days when Sara sits on the opposite aisle seat.

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Blacksburg, VA, Aug. 28, 2013

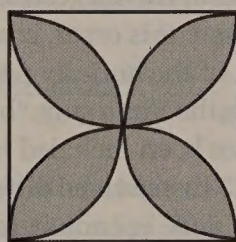
A MIDDLE SCHOOL PROBLEM BECOMES AN ALGEBRA PROBLEM

The Palette of Problems in the September 2013 issue of *Mathematics Teaching in the Middle School* contains an excellent problem suitable for beginning algebra students, no matter what grade.

Problem 14, Palette of Problems, MTMS

Problem 14 (MTMS vol. 19, no. 2, p. 83) states:

Four semicircles are drawn within a square intersecting at a single point equidistant from each side of the square as shown in the figure. The overlapping sections are shaded. What percentage of the square is shaded?



Label the area of each petal as B and the area of each of the other 4 sections as A . If the square has side length 2, then $4A + 4B = 4$, so $A + B = 1$. Also, $A + 2B = (1^2)\pi/2 = \pi/2$. Subtraction gives $B = \pi/2 - 1$. The percentage required is $4B/4 = B$, approximately 0.57, or 57%.

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Editor's note: This problem is similar to problem 12 for the February 2014 MT Calendar (MT Feb. 2014, vol. 107, no. 6, p. 440).

THOUGHTS ON $\sqrt{x^2 + 9}$

Why do so many students in classes from algebra to calculus rewrite expressions such as $\sqrt{x^2 + 9}$ as $x + 3$? Is this simply a mistake made in the heat of problem solving?

I have found that order-of-operations problems involving grouping symbols lie at the heart of many common mistakes that students make during symbolic manipulation, even students who think carefully about what they are doing. For example, errors involving an expression such as $x/(x + 3)$ may have to do with grouping (among other misconceptions). Students may not be aware that the fraction bar not only indicates division but also is a grouping symbol. The same is true for an expression such as $\sqrt{x^2 + 9}$.

One natural source of the problem comes from the acronym for order of operations: PEMDAS. Since the P stands for "parentheses," the fact that there may be other grouping symbols is lost in translation. In the particular case of the radical, by replacing roots with appropriate exponential notation, we can bring back the parentheses. This doesn't help with $x/(x + 3)$. In any case, even with a broader notion of grouping symbols, students still make the same errors. We can replace PEMDAS with GEMDAS (with G being "grouping symbol") or with GEMS (recognizing that certain operations are on the same level), but this algorithm still does not help students. Why not?

When I asked a highly motivated student to write a list of instructions for evaluating the expression $\sqrt{x^2 + 9}$, she responded by writing "take the square root" as the first step. She defended her claim vigorously. Her justification was that if the radical represented a grouping symbol and grouping symbols come first in the order of operations, then we should do what the radical tells us to do first.

This justification gets at an issue subtly buried in the order of operations that had not occurred to me up to that point. Whether you use PEMDAS, GEMDAS, or GEMS, the grouping symbol piece is the only step in the list of operations that itself does not involve an action. Parentheses themselves do not act. However, symbols such as the radical entail

an action; they tell us to do something. If they are grouping symbols and if grouping symbols come first, then we should “do” what the symbol tells us to do before anything else.

A symbol such as a radical plays two roles. On the one hand, it tells us to perform an action—take the positive square root. On the other hand, it functions as a grouping symbol—specifically telling us the order in which to do the operations. These two roles must be separated to understand fully the meaning of the expression. The grouping role tells us that we should perform the operations enclosed by the radical symbol first. Once all these operations have been completed, we apply the action represented by the symbol. In a way, a symbol with these two roles postpones its own action until after everything enclosed has been performed.

This subtlety is fundamental to success with algebra. Such a notion should be addressed explicitly with students but carefully scaffolded. The basic ideas of order of operations are taught early in elementary schools in the context of numerical computations. Generalization occurs later when symbolic algebra is introduced. At this later stage, when symbols such as the radical are introduced, teachers should address this issue. This discussion can be combined with sense-making activities such as deconstructing formulas into a list of instructions.

Would it make a difference? I don’t claim to know, but let’s try it and see!

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Big Rapids, MI, Sept. 12, 2013

PROBLEMS 14, 27, AND 30, SEPTEMBER 2013 CALENDAR

Regarding problem 14 in the September 2013 Calendar (MT Sept. 2013, vol. 107, no. 2, p. 120): If $f(x) = 3x^2 + bx - 20$, by the factor theorem $f(4) = 0$ leads to $b = -7$.

Problem 27 (MT Sept. 2013, vol. 107, no. 2, p. 122) states:

Four identical-looking objects weigh 3, 5, 8, and 11 oz., respectively. Using a balance scale, how can you deter-

mine the heaviest object in just two weighings?

There are only three possible weighings balancing two objects on each side: (3, 5) and (8, 11); (3, 8) and (5, 11); and (3, 11) and (5, 8). In all three cases, one pair is heavier than the other; they are (8, 11), (5, 11), and (3, 11), respectively. The heavier pair always contains the heaviest object, which is 11 oz. Now, weighing individual objects of the heavier pair determines the heaviest one.

Regarding problem 30 (MT Sept. 2013, p. 121), the first line of the solution should read as follows: “The area of a triangle is *half* the product . . .

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Madison, WI, Sept. 10, 2013

COLORADO LOTTERY

I enjoyed “Boulder Lottery” (Media Clips, MT Oct. 2013, vol. 107, no. 3, pp. 172, 174), which is ripe for mathematical analysis. However, although the question writers correctly take the author to task for misusing vocabulary (*probability* vs. *odds*), they overlook a fundamental factual error: The lump sum payment is *not* “after taxes,” as claimed. Indeed, the suggested solutions perpetuate this error, concluding that the sum of the annuity payments is larger than the lump sum “because taxes have not been included in the calculations.” Instead, the difference reflects the simple economic fact that money today is worth more than money tomorrow.

This is a mystery of compound interest, not taxes. The truth about taxes appears on the Colorado Lottery website (<http://www.coloradolottery.com/winner-section/winners/claiming-prizes/cash-vs-annuity/>). By law, the lottery withholds 29% of the winnings for taxes (or 34% if the winner does not provide a Social security number). Neither of the amounts quoted in the article includes taxes. To be sure, we can consider some additional facts. Again referring to the Colorado Lottery website (<http://www.coloradolottery.com/winner-section/winners/financial-counsel/>), we learn that the first annuity installment is fixed at 2.5% of the jackpot and increases by 3.7% per year thereafter, whereas the lump sum payment is approximately half the jackpot. In this case, the jackpot was \$5.3 million (mentioned in the headline of the original article; also computed by multiplying the initial installment of \$133,300), and the lump sum option is \$2.6 million, or approximately half the jackpot.

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AN UNEXPECTED EQUATION

A standard problem in trigonometry requires determining the height, h , given the angle of elevation from each of two points that are a given distance, y , apart in the same plane (see **fig. 1 [Metz]**).

To find h , we use the two ratios $\tan(\angle DBC) = h/x$ and $\tan A = h/(x + y)$. So $x = h/\tan(\angle DBC)$ and also $x = (h/\tan A) - y$. Thus, $h/\tan(\angle DBC) =$

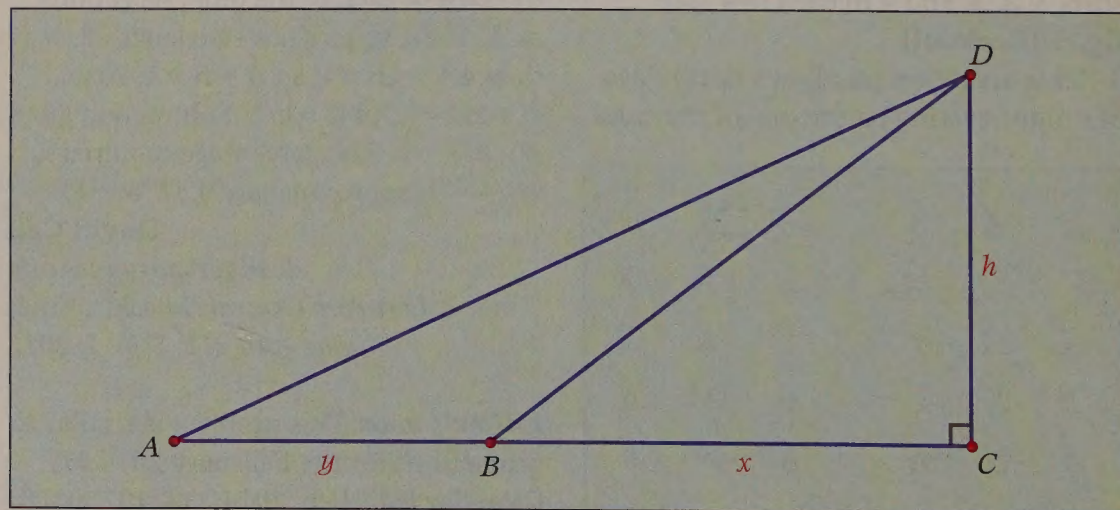


Fig. 1 (Metz)

$(h/\tan A) - y$, making the solution to the general problem this:

$$h = \frac{y}{\frac{1}{\tan A} - \frac{1}{\tan(DBC)}}$$

The last equation can be written as

$$\frac{1}{\tan A} - \frac{1}{\tan(DBC)} = \frac{1}{h/y}$$

Returning to the original ratios, we then have

$$\frac{1}{h/(x+y)} = \frac{1}{h/x} + \frac{1}{h/y}$$

From this final equation, we have the trivial fact that $(x+y)/h = x/h + y/h$. However, it fits the familiar $1/c = 1/a + 1/b$ formula that has so many applications—an interesting result.

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PYTHAGOREAN TRIPLE GENERATOR

There have been many different Pythagorean triple generators developed in the past, and a new one might seem superfluous. However, those developed in the past do not generate all primitive Pythagorean triples. (A *primitive Pythagorean triple* is a Pythagorean triple where the three numbers do not collectively share any factors other than 1.) This algorithm does, in fact, generate all primitive Pythagorean triples for one of the triangle legs of given length A .

We let A denote the natural number that we would like to use to generate Pythagorean triples. Fa denotes the factors of A .

When A is odd, the primitive Pythagorean triples will be generated with this formula:

$$\left\{ A, \frac{A^2 - (Fa)^4}{2(Fa)^2}, \frac{A^2 + (Fa)^4}{2(Fa)^2} \right\}$$

Let's say, for example, that we want to find all Pythagorean triples with the number 15. That is, $A = 15$. First, list all of the factors of 15, which are 1, 3, 5, and 15.

Then plug each of those into the formulas to generate all of the triples (something much more easily done using a computer).

First, let $Fa = 1$:

$$\left\{ 15, \frac{15^2 - 1^4}{2 \cdot 1^2} = 112, \frac{15^2 + 1^4}{2 \cdot 1^2} = 113 \right\}$$

Next, $Fa = 3$:

$$\left\{ 15, \frac{15^2 - 3^4}{2 \cdot 3^2} = 8, \frac{15^2 + 3^4}{2 \cdot 3^2} = 17 \right\}$$

Then $Fa = 5$:

$$\left\{ 15, \frac{15^2 - 5^4}{2 \cdot 5^2} = -8, \frac{15^2 + 5^4}{2 \cdot 5^2} = 17 \right\}$$

Finally, $Fa = 15$:

$$\left\{ 15, \frac{15^2 - 15^4}{2 \cdot 15^2} = -112, \frac{15^2 + 15^4}{2 \cdot 15^2} = 113 \right\}$$

So we get $\{15, 112, 113\}$ and $\{8, 15, 17\}$. Notice that when you use the smaller factors you get the correct triples, but the larger numbers give you negative values. You only need to plug in the first half of the list of factors using $(Fa)^2 < A$. So, to find the triples for 21, you would only need to use 1 and 3. The factors 7 and 21 would be superfluous.

The algorithm for even numbers is a little different. The formula becomes

$$\left\{ A, \frac{A^2 - 4(Fa)^2}{4(Fa)}, \frac{A^2 + 4(Fa)^2}{4(Fa)} \right\}$$

Let's use the number 20 as the example. Its factors are 1, 2, 4, 5, 10, and 20.

First, $Fa = 1$:

$$\left\{ 20, \frac{20^2 - 4(1)^2}{4 \cdot 1} = 99, \frac{20^2 + 4(1)^2}{4 \cdot 1} = 101 \right\}$$

Next, use $Fa = 2$:

$$\left\{ 20, \frac{20^2 - 4(2)^2}{4 \cdot 2} = 48, \frac{20^2 + 4(2)^2}{4 \cdot 2} = 52 \right\}$$

Then $Fa = 4$:

$$\left\{ 20, \frac{20^2 - 4(4)^2}{4 \cdot 4} = 21, \frac{20^2 + 4(4)^2}{4 \cdot 4} = 29 \right\}$$

Then $Fa = 5$:

$$\left\{ 20, \frac{20^2 - 4(5)^2}{4 \cdot 5} = 15, \frac{20^2 + 4(5)^2}{4 \cdot 5} = 25 \right\}$$

Finally, we use $Fa = 10$:

$$\left\{ 20, \frac{20^2 - 4(10)^2}{4 \cdot 10} = 0, \frac{20^2 + 4(10)^2}{4 \cdot 10} = 20 \right\}$$

Using the factor 20 itself does not give a new triple because $A^2 - 4A^2$ in the numerator will result in a nonpositive integer. The list of Pythagorean triples we get using $A = 20$ are $\{20, 99, 101\}$, $\{20, 48, 52\}$, $\{20, 21, 29\}$, and $\{15, 20, 25\}$.

For a given leg with even length A , the algorithm generates all the Pythagorean triples, whereas for odd A , it generates the primitive ones.

The weakness of the proposed algorithm lies in the fact that it does generate some false triples. Those triples, however, are easily recognizable, since they will include a side length of zero or a negative number. We would just have to eliminate those solutions and use only the ones with positive numbers.

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New York City, Sept. 13, 2013

Editor's note: The process does not generate primitive triples only. An example using $A = 27$ and $Fa = 3$ yields the Pythagorean triple $\{27, 36, 45\}$, which is not primitive.

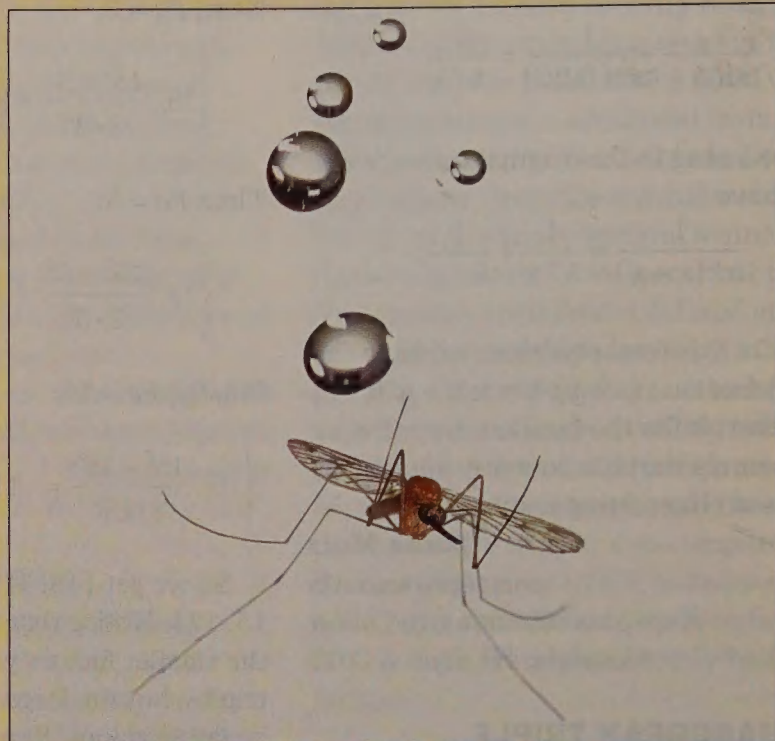
more4U

For a proof of Greenblatt and Koplon's work, download one of the free apps for your smartphone and then scan this tag to access www.nctm.org/mt054.



Mosquitoes

Imagine how tough life would be if raindrops weighed 3 tons apiece as they fell out of the sky at 20 mph. That's how raindrops look to a mosquito, yet a raindrop weighing 50 times more than one can hit the insect and the mosquito will survive. . . .



SOURCE: [HTTP://DICKERSON.GATECH.EDU/FILE/MOSQUITOES.HTML](http://dickerson.gatech.edu/file/mosquitoes.html)

Source: "Splish Splat? Why Raindrops Don't Kill Mosquitoes," NPR's *Morning Edition*, June 5, 2012, <http://www.npr.org/2012/06/05/154300189/splish-splat-why-raindrops-dont-kill-mosquitoes>

The following questions help make sense of these comparisons.

1. An average-sized raindrop weighs about 4 mg. In comparison, most mosquitoes weigh in at about 2.5 mg. How many times bigger is a 4 mg raindrop than a 2.5 mg mosquito?
2. The clip mentions 3 tons. How many pounds are in 3 tons? What would be the corresponding mass in kilograms?
3. How many times heavier is a 3-ton raindrop than a person who weighs 70 kg?
4. The density of water is roughly 1000 kg/m^3 .

(a) Use this information to estimate the volume of the 3-ton raindrop in cubic meters.

(b) If we assume that the raindrop is a sphere, what is the radius in meters?

5. An average-sized raindrop is 2 mm in diameter, weighs 4 mg, and falls at about 9 m/sec (see the physics fact-book at hypertextbook.com/facts).

(a) How much heavier is the 3-ton raindrop than the average-sized raindrop?

(b) How much heavier is the 70-kg person than the average-sized mosquito?

6. Does your analysis support the first statement from the clip?

7. To find out how the mosquitoes survive, follow the link at the bottom of the clip.

Media Clips appears in every issue of *Mathematics Teacher*, offering readers contemporary, authentic applications of quantitative reasoning based on print or electronic media. All submissions should be sent to the editors. For information on the department and guidelines for submitting a clip, visit <http://www.nctm.org/publications/content.aspx?id=10440#media>.

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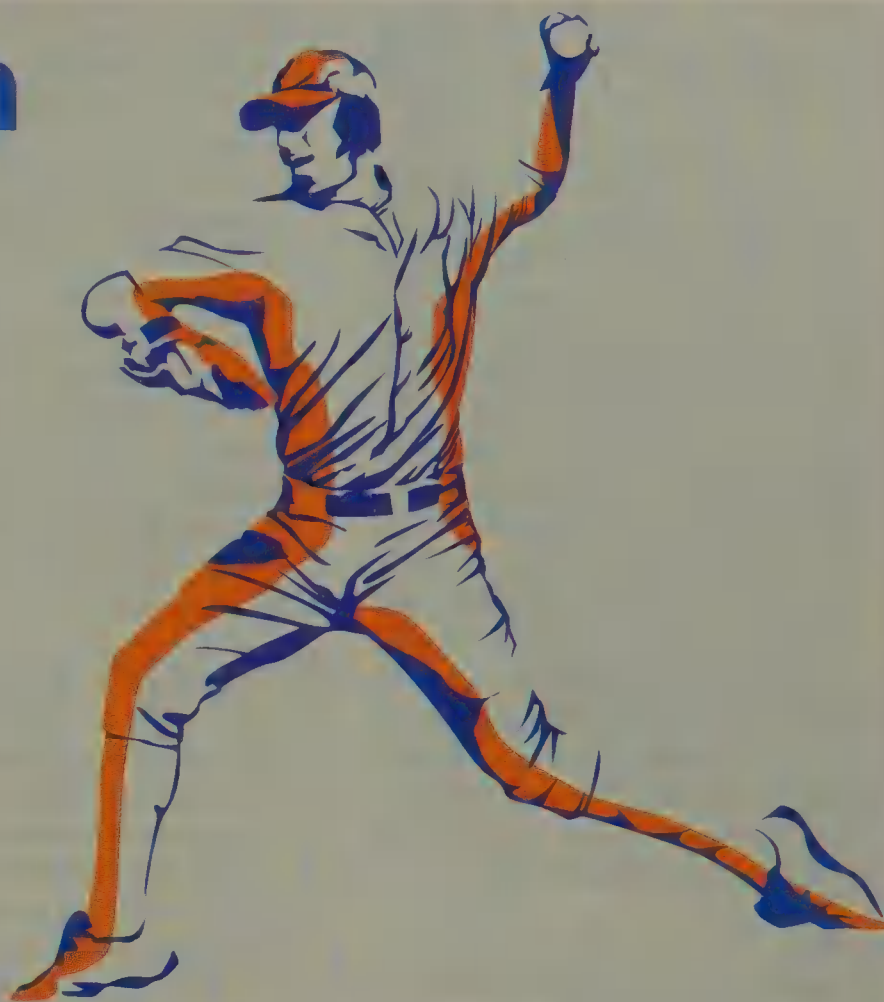
Pythagorean Baseball

National League first-round previews

D-backs vs. Brewers

Regular season records: Diamondbacks 94-68 (NL West champions), Brewers 96-66 (NL Central champions).

Why the Diamondbacks can win: If you seeded the field, one through eight, the D-backs would get the eighth seed. They outscored foes by only 69 runs, for a “Pythagorean” record of 88-74. Their strength is a bullpen that allowed the lowest opponents’ on-base percentage (.299) in the game, and an offense that has both power (the third-most homers in the NL) and speed (second-most stolen bases).



MELLIOTT1605/THINKSTOCK

Source: Dave Sheinin, “National League First-Round Previews,” Sept. 30, 2011, http://www.washingtonpost.com/sports/national-league-first-round-previews/2011/09/30/gIQATsoDBL_story.html

- As the clip states, the Arizona Diamondbacks were the 2011 National League West champions with a record of 94 wins and 68 losses.

- What was the team’s winning percentage? (Winning percentage equals wins divided by games played and is typically reported as a three-place decimal, not as a true percentage.)
- What was the winning percentage of the Diamondbacks’ playoff opponent, the Milwaukee Brewers?

- According to Bill James, author of the *Baseball Abstract*, an accurate predictor of a team’s winning percentage is the “Pythagorean theorem of baseball”:

Pythagorean winning percentage

$$= \frac{(\text{runs scored})^2}{(\text{runs scored})^2 + (\text{runs allowed})^2}$$

In 2011, the Brewers scored 721 runs and allowed 638.

- What was the Brewers’ Pythagorean winning percentage?
- What win-loss record for a 162-game season would correspond to this Pythagorean winning percentage?
- How does this record compare with the Brewers’ actual record?

- If the number of Pythagorean wins divided by the total number of games played equals the Pythagorean winning percentage, show that

$$\frac{\text{Pythagorean wins}}{\text{Pythagorean losses}} = \frac{(\text{runs scored})^2}{(\text{runs allowed})^2}$$

- The article states that the Diamondbacks outscored their opponents by 69 runs. The stated Pythagorean record of 88-74 was calculated using an exponent of 1.83 instead of 2 throughout the “Pythagorean” formulas, as is now often done by sabermetricians. According to our formula, the Diamondbacks’ Pythagorean record would be 89-73. Use our formula with an exponent of 2 to determine a possible value for the number of runs allowed that would lead the Diamondbacks to 89 Pythagorean wins in the season of 162 games.

- Changing the number of runs slightly will still result in rounding to 89 Pythagorean wins. Find a range of values for the number of runs allowed that will result in a Pythagorean record rounded to 89-73. Explain your thought process, keeping in mind that runs are whole numbers.

"Mosquitoes" answers

1. A raindrop is $4/2.5$, or 1.6, times as big as a mosquito.
2. Since 1 ton is 2000 lbs., 3 tons is 6000 lbs. A kilogram corresponds to approximately 2.20462 lbs. Dividing 6000 by 2.20462 shows that there are about 2721.5575 kg in 3 tons.
3. Using 2721.6 kg for the raindrop, we find that $2721.6/70 \approx 38.879$. Thus, a 3-ton raindrop is about 40 times heavier than a person who weighs 70 kg. Alternately, the 70-kg person weighs $70 \cdot 2.20462 \approx 154.3234$ lbs. Again, the ratio $6000/154.3234 \approx 38.879$, or approximately 40.
4. (a) The raindrop would be approximately 2721.6 kg. Dividing the mass by the density gives us the volume $2721.6/1000 = 2.7216 \text{ m}^3$ of water.

(b) The volume of a sphere is related to the radius by the equation $V_{\text{sphere}} = (4/3)\pi r^3$. We can solve for the radius as $r = \sqrt[3]{3V/(4\pi)}$. Thus, $r = 0.866 \text{ m}$ for this 3-ton raindrop.
5. (a) The 3-ton raindrop weighs in at 2,721.6 kg. An average raindrop is 4 milligrams (mg). One milligram is one thousandth of a gram, and a kilogram is 1,000 grams. We can do dimensional analysis to get $2,721.6 \text{ kg} \times 1000 \text{ (g per kg)} \times 1000 \text{ (mg per g)}$ to get $2,721,600,000 = 2.7216 \times 10^9 \text{ mg}$ in the giant raindrop. Dividing by 4 tells us that the 3-ton raindrop is $2.7216/4 \times 10^9$, or 6.804×10^8 . That giant raindrop is $2.7216/4 \times 10^9 = 680,400,000$, or almost 700 million times as heavy as a typical raindrop.

(b) Using the same analysis, we find that the person who weighs 70 kg is $70 \times 1000 \times 1000$, or $7.0 \times 10^7 \text{ mg}$. The mosquito is

about 2.5 mg (from question 1). Thus, the person is $7/2.5 \times 10^7$, or 28 million times as big.

6. To compare 20 mph and 9 m/s, we need the conversions: 1.60934 kilometers in a mile, 1000 meters in a kilometer, and 3600 seconds in an hour. So 20 mph is approximately 8.9408 m/s. This calculation supports the 20 mph mentioned in the clip.

A 4-mg raindrop is about 1.6 times as heavy as a mosquito, but the 3-ton raindrop is about 40 times as heavy as a 70-kg person. This is not a reasonable comparison. Raindrops range in size from 0.004 mg in a drizzle to 300 mg in a heavy storm (see <http://hypertextbook.com/facts/1999/MichaelKodransky.shtml>).

7. A sneak peek at the answer: The mosquito merges briefly with the drop before the air drag against the wings pulls it free again.

"Pythagorean Baseball" answers

1. (a) Because the number of games played equals wins plus losses, Arizona played $94 + 68 = 162$ games, and its winning percentage was $94/162 \approx 0.580$.

(b) Milwaukee's winning percentage was $96/162 \approx 0.593$.

2. (a) Milwaukee's Pythagorean winning percentage was

$$\frac{(\text{runs scored})^2}{(\text{runs scored})^2 + (\text{runs allowed})^2} = \frac{(721)^2}{(721)^2 + (638)^2} = \frac{519,841}{926,885} \approx 0.561.$$

- (b) We would expect the team to win $0.561 \cdot 162 \approx 91$ games and lose $162 - 91 = 71$ games. Thus, the Brewers' predicted record would be 91-71.

- (c) The Brewers' actual record of 96-66 was 5 wins better than their predicted Pythagorean record of 91-71. In baseball-statistical sabermetric lingo, we would say that the Brewers used their runs "efficiently."

3. Because the wins and losses must total the number of games a team plays,

$$\begin{aligned} & \frac{\text{Pythagorean wins}}{\text{total number of games played}} \\ &= \frac{\text{Pythagorean wins}}{\text{Pythagorean wins} + \text{Pythagorean losses}} \\ &= \frac{1}{1 + \frac{\text{Pythagorean losses}}{\text{Pythagorean wins}}} \end{aligned}$$

But we also know that

Pythagorean winning percentage

$$\begin{aligned} &= \frac{(\text{runs scored})^2}{(\text{runs scored})^2 + (\text{runs allowed})^2} \\ &= \frac{1}{1 + \frac{(\text{runs allowed})^2}{(\text{runs scored})^2}} \end{aligned}$$

Therefore,

$$\frac{\text{Pythagorean losses}}{\text{Pythagorean wins}} = \frac{(\text{runs allowed})^2}{(\text{runs scored})^2}$$

establishes the desired proportional relationship.

4. We let a represent the number of runs allowed, so $a + 69$ would be the number of runs scored, and then solve

$$\frac{89}{73} = \frac{(a + 69)^2}{a^2}.$$

Cross multiplication yields a quadratic equation that can be rewritten as $0 = 16a^2 - 10,074a - 347,553$. Solving this equation produces

runs-allowed values of about -32.792 and 662.417 . Disregard the negative solution and expect that values of a around 662 would yield 89 wins.

- If we multiply the Pythagorean winning percentage by 162 and round the product, we will have the number of Pythagorean wins, y_2 . We write the following formulas, where x represents the number of runs allowed and y_1 the Pythagorean winning percentage—

$$y_1 = \frac{(x + 69)^2}{x^2 + (x + 69)^2} \text{ and}$$

$$y_2 = \text{round}(162 \cdot y_1, 0)$$

—and use the table feature of a calculator to examine the results. A range of x -values in the neighborhood of

662 results in the output y_2 equal to 89. We find that whole-number runs allowed in the interval $[622, 709]$ will work. The corresponding numbers of runs scored would range from 691 through 778, inclusive. The actual record for the Diamondbacks was 731 runs scored and 662 runs allowed, both within the range calculated.

Alternate solution: Let $f(a) = (a + 69)^2/a^2$. Because wins, losses, and runs are all whole numbers, we seek values of $f(a)$ such that

$$1.20408 \approx \frac{88.5}{73.5} \leq f(a) \leq \frac{89.5}{72.5} \approx 1.23448.$$

Scrolling through a table of values for $f(a)$ reveals that $622 \leq a \leq 709$ and, thus, $691 \leq a + 69 \leq 778$.



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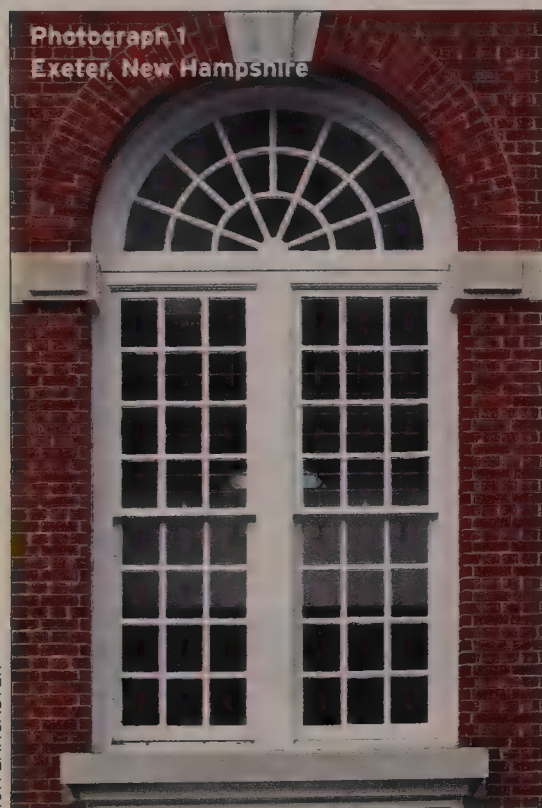
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Seeing the Light



The term *Norman architecture* is used to categorize styles of Romanesque architecture developed by the Normans in northwestern Europe, particularly England, in the eleventh and twelfth centuries. The Normans introduced castles, fortifications, monasteries, abbeys, churches, and cathedrals, all with characteristic rounded arches, particularly over windows and doorways, and massive proportions.

1. Norman windows consist of a rectangle topped by a semicircle, as shown in **photographs 1–4**. Suppose that the perimeter of a Norman window is 8 m. Let the radius of the semicircle be x m, let the height of the rectangle be h m (see **fig. 1**), and let the area of the window be y m².

- (a) If $x = 1$, find the value of h .
- (b) Suppose that x decreases in value from 1. Describe how the value of h will change. Describe how the shape of the window will change.
- (c) Suppose instead that x increases. Describe how the value of h will change and how the shape of the window will change.
- (d) Determine h as a function of x . Find the domain of this function. Sketch the graph of h versus x . Is the graph consistent with your answers from parts (b) and (c)?

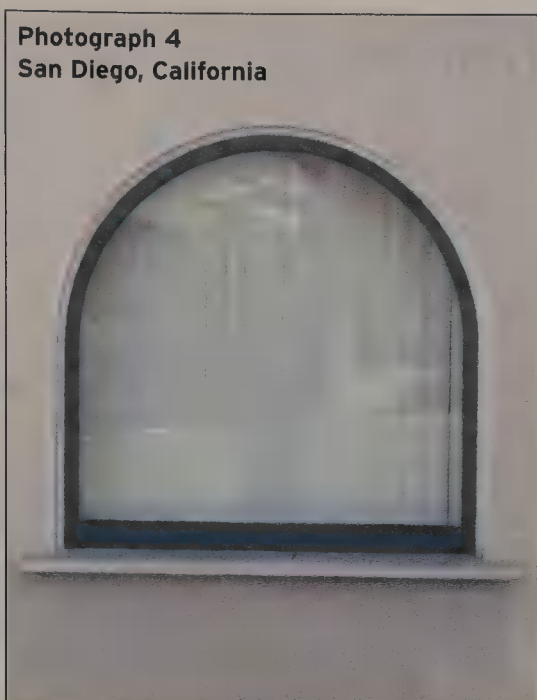
Mathematical Lens uses photographs as a springboard for mathematical inquiry and appears in every issue of *Mathematics Teacher*. All submissions should be sent to the department editors. For more background information on Mathematical Lens and guidelines for submitting a photograph and questions, please visit <http://www.nctm.org/publications/content.aspx?id=10440#lens>.

Edited by **Ron Lancaster**, ron2718@nas.net, University of Toronto, ON, Canada, and **Brigitte Bentele**, brigitte.bentele@trinityschoolnyc.org, Trinity School, New York, NY

Photograph 3
Raleigh, North Carolina



Photograph 4
San Diego, California



- (e) Show that the area y equals

$$\frac{-(4 + \pi)x^2 + 16x}{2}.$$

Use a graphing utility to sketch the graph of y versus x . Find the approximate value of x for which y is a maximum.

- (f) The exact value of x for which y is a maximum can be found by completing the square for this quadratic expression, but the procedure will be tedious because the coefficients are not integers. Instead, rewrite the equation for y in terms of x in factored form to find the exact value of x for which y is a maximum. Is this value consistent with the approximate value found in part (e)?

- (g) In part (f), you should have found that the value of y is a maximum when $x = 8/(4 + \pi)$. Find the corresponding value of h and sketch this Norman window.

- (h) In the Norman window with maximal area, find the ratio of the height of the rectangle to the base of the rectangle. Which of the windows in **photographs 1, 2, 3, and 4** has a height-to-base ratio closest to this ratio?

- Suppose that the perimeter of a Norman window is P . Show that the area of the window is as large as possible when the radius of the semicircle and the height of the rectangle are both equal to $P/(4 + \pi)$.
- Suppose that the perimeter of a Norman window is $8 + 2\pi$ meters. Use the results for question 2 to state the dimensions of the window that has the largest area.
- Consider a version of the Norman window that has a semicircle cut out of the window instead of sitting on top of it (see **fig. 2**). Suppose that the perimeter of the Norman window is once again 8 m. Find the dimensions of the window for which the area y is as large as possible.

- Figure 3** shows a version of a Norman window for which the semicircle has been replaced by a semi-ellipse, with the length of the major axis (which is parallel to the rectangle's base) equal to twice the length of the minor axis. Let the length of the semimajor axis be x m, let the height of the rectangle be h m, and let the area of the window be y m². Suppose that the perimeter of the window is equal to 8 m. What formulas are needed to determine h and y as functions of x and to determine the dimensions that would produce the largest window?

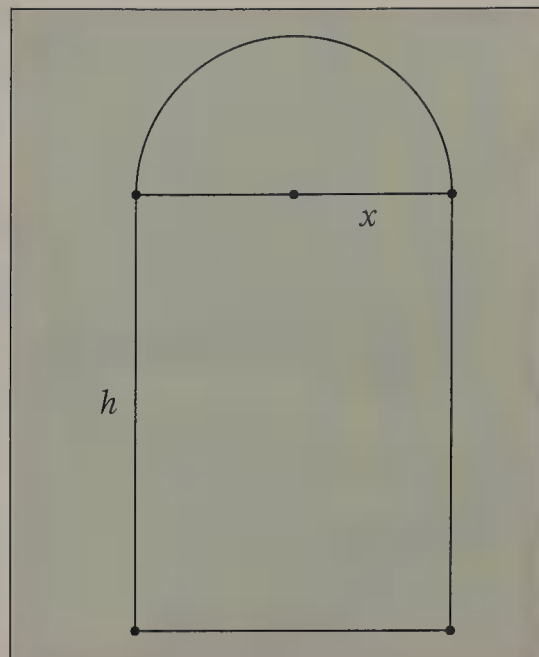


Fig. 1 This Norman window has a rectangle of dimensions $2x$ and h surmounted by a semicircle with radius x .

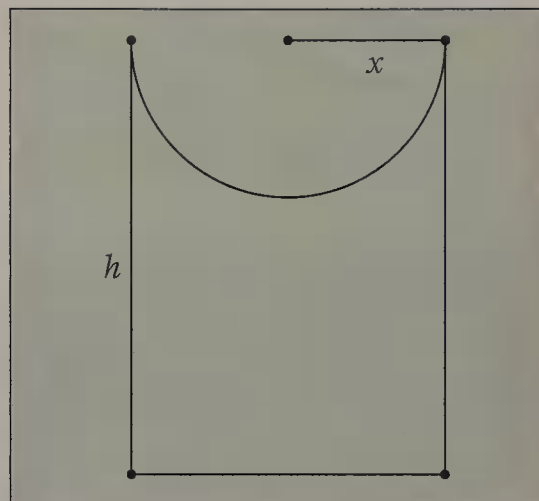


Fig. 2 In this variation, the semicircle is cut out of the rectangle instead of surmounted above it.

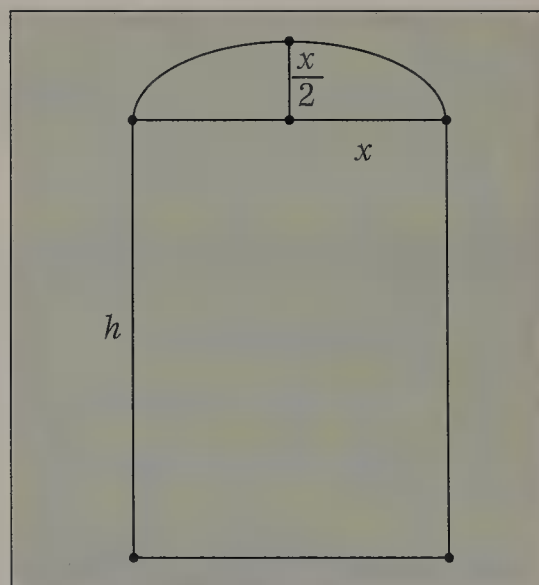


Fig. 3 Here the rectangle is surmounted by a semi-ellipse with major and minor axes in the ratio of 2:1.

MATHEMATICAL LENS

solutions

1. (a) We can substitute $x = 1$ into an equation for the perimeter of the window:

$$2h + 2x + \pi x = 8$$

leads to

$$h = \frac{6 - \pi}{2} \approx 1.43.$$

- (b) When the value of x decreases from 1, the value of h will increase. Since x represents the length of the semicircle, its lower bound is 0. The window will become narrower at the base and taller (see **fig. 4** for examples that show the transition).

- (c) When the value of x increases, the value of h will decrease. Since h represents the height of the window, its lower bound is 0. Therefore, there is an upper limit to the size of x . The window will become wider at the base and shorter (see **fig. 5** for examples that show the transition).

- (d) We start with an equation for the perimeter and solve for h :

$$2h + 2x + \pi x = 8 \rightarrow$$

$$h = \frac{8 - (2 + \pi)x}{2}.$$

Because $h \geq 0$, we impose

$$0 \leq x \leq 8/(2 + \pi) \leq 1.56.$$

The graph of h versus x , shown in **figure 6**, shows the directed change and bounds from parts (b) and (c).

- (e) The area of the window is the sum of the areas of the rectangle and the semicircle:

$$\begin{aligned} y &= 2xh + \left(\frac{1}{2}\right)\pi x^2 \\ &= 2x \left(\frac{8 - (2 + \pi)x}{2} \right) + \frac{\pi x^2}{2} \\ &= \frac{16x - 4x^2 - 2\pi x^2 + \pi x^2}{2} \\ &= \frac{-(4 + \pi)x^2 + 16x}{2} \end{aligned}$$

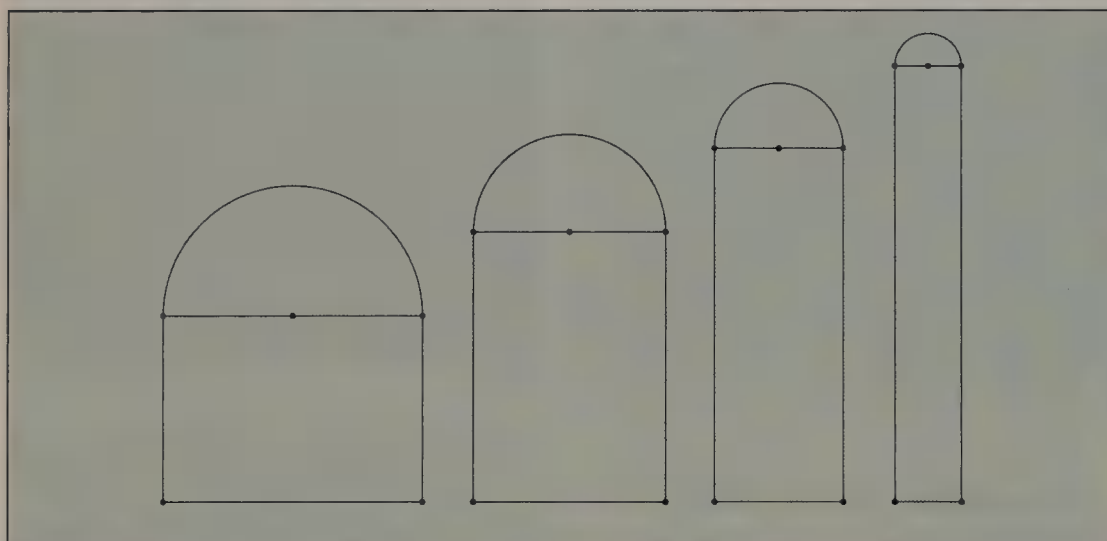


Fig. 4 For constant perimeter, as the width decreases, the window becomes taller and narrower.

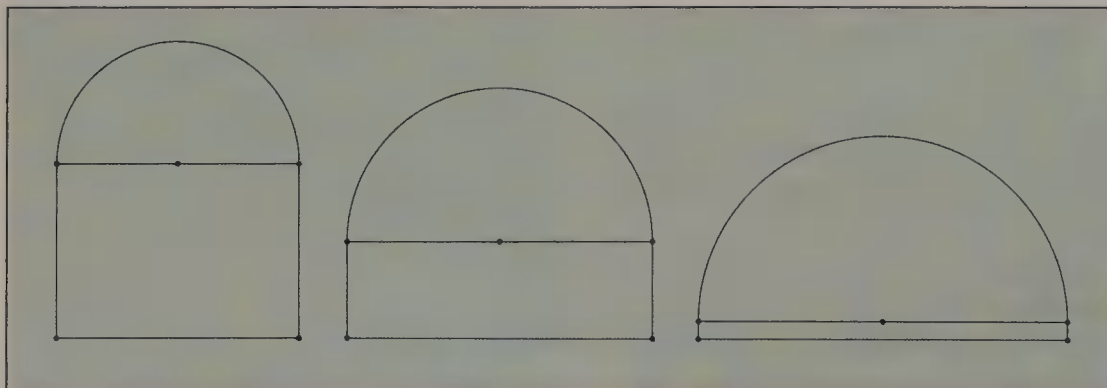


Fig. 5 For constant perimeter, as the width increases, the window becomes shorter and more squat.

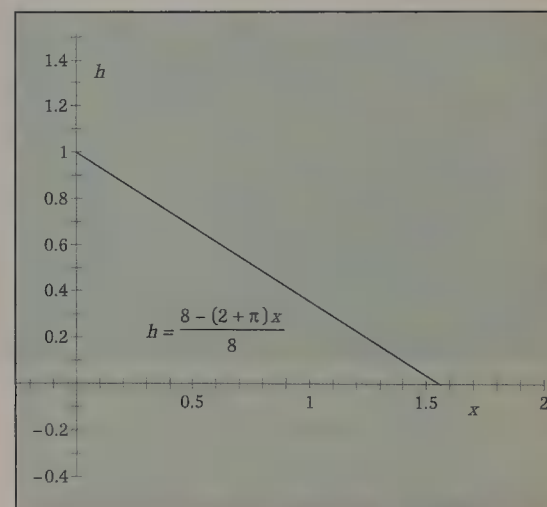


Fig. 6 The values of x and h are bounded.

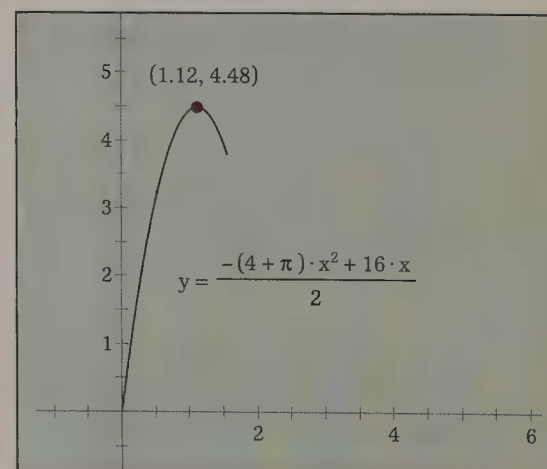


Fig. 7 The maximum area can be seen graphically.

Figure 7 shows the graph of y versus x and the coordinates of the maximum point. The value of y is a maximum when x is approximately equal to 1.12.

- (f) The vertex of a parabola occurs midway between two x -intercepts. Thus, if $y = k(x - a)(x - b)$, the extreme value of y occurs when $x = (a + b)/2$. Factoring yields

$$\frac{-(4 + \pi)x^2 + 16x}{2} = -\left(\frac{1}{2}\right)x[(4 + \pi)x - 16].$$

Thus, $a = 0$, $b = 16/(4 + \pi)$, and $(a + b)/2 = 8/(4 + \pi) \approx 1.12$. This result is consistent with the approximate value of x found in part (e).

- (g) Using $x = 8/(4 + \pi)$ from part (f) in the equation from part (d), we have

$$\begin{aligned} h &= \frac{8 - (2 + \pi)x}{2} \\ &= \frac{8 - (2 + \pi)\frac{8}{4 + \pi}}{2} \\ &= \frac{8(4 + \pi - 2 - \pi)}{2(4 + \pi)} \\ &= \frac{8}{4 + \pi}. \end{aligned}$$

This calculation shows that in the Norman window with maximum area, the height of the rectangle is the same as the radius of the semi-

circle, meaning that the semicircle is situated on top of two identical squares (see **fig. 8**).

- (h) In the rectangle, the ratio of height to the base is 1:2. The Norman window in San Diego (see **photograph 4**) is closest to this ratio.

- Students studying calculus can answer this question using differentiation. Alternatively, students can use the procedure in part (f) to show that the value of y is a maximum when $x = h = P/(4 + \pi)$.
- Since $(8 + 2\pi)/(4 + \pi) = 2$, it follows that the radius of the semicircle is 2 m and the height of the rectangle is 2 m.
- The equation for area becomes

$$\begin{aligned} y &= 2xh - \frac{\pi x^2}{2} \\ &= 2x\left(\frac{8 - (2 + \pi)x}{2}\right) - \frac{\pi x^2}{2} \\ &= \frac{16x - 4x^2 - 2\pi x^2 - \pi x^2}{2} \\ &= \frac{-(4 + 3\pi)x^2 + 16x}{2}. \end{aligned}$$

Using the same procedure as in part (f) or using calculus, we can show that the value of y is a maximum when $x = 8/(4 + 3\pi)$ and $h = (8 + 8\pi)/(4 + 3\pi)$.

For additional enrichment, students can use the arc length function in a CAS program to attempt to solve the problem in question 5.

- Students will have to use the formulas for the perimeter and the area of a rectangle and an ellipse. If the semimajor and semiminor axes of the ellipse are a and b , the area of the ellipse is equal to πab . Most students will be surprised that a search for a formula for the perimeter of an ellipse leads to a dead end. There is no closed formula for the perimeter of an ellipse.

Some students might know or guess that the solution to the Norman window problem in question 1 is a semi-

circle on top of two squares. If so, teachers might want to assign question 4, in which the answer is not as obvious. Teachers might also want to assign a variation of the problem in which the semicircle is tinted and allows only half the light through. In this case, the goal is to find the dimensions of the window for which the most light passes through.

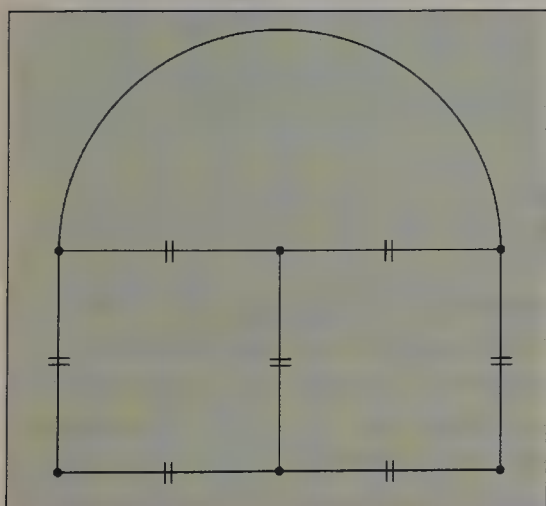


Fig. 8 The Norman window with maximum area has the radius of its semicircle equal to the height of its rectangle.

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Mathematics Programs as the Core for Student Learning

Daniel Brahier, Steve Leinwand, and DeAnn Huinker

The National Council of Teachers of Mathematics (NCTM) launched the “standards-based” education movement in North America in 1989 with the release of *Curriculum and Evaluation Standards for School Mathematics*, an unprecedented action to promote systemic improvement in mathematics education. Now, twenty-five years later, the widespread adoption of the Common Core State Standards for Mathematics (CCSSM) by forty-five states provides an opportunity to reenergize and focus our commitment to significant improvement in mathematics education (CCSSI 2010).

The need for coherent standards has been endorsed by all states and provinces, even by those who have not adopted the CCSSM. As NCTM has stated, “The CCSSM offers a foundation for the development of more rigorous, focused, and coherent mathematics curricula, instruction, and assessments that promote conceptual understanding and reasoning as well as skill fluency. This foundation will help ensure that all students are ready for college and the workplace when they graduate from high school and that they are prepared to take their place as productive, full participants in society” (NCTM 2013).

However, we must remember that standards

do not teach; teachers teach. The CCSSM focus and clarify common outcomes for student learning and promote coherence across grades through mathematics learning progressions. They provide motivation for the development of new instructional resources and assessments but do not tell teachers, coaches, administrators, or policymakers what to do at the classroom, school, and district levels or how to begin making essential changes to implement these Standards. Thus, NCTM’s new *Principles to Actions: Ensuring Mathematical Success for All* (NCTM 2014) provides direction in filling the gap between the adoption of the CCSSM and the enactment of policies and programs required for its widespread and successful implementation. The overarching message conveyed in *Principles to Actions* is that effective teaching is the non-negotiable core to ensure that all students learn mathematics and that such teaching requires a range of actions at the state or provincial level as well as at the district, school, and classroom levels.

UPDATED PRINCIPLES

In *Principles to Actions*, NCTM sets forth recommended actions for education leaders and policy-

Guiding Principles for School Mathematics

Teaching and Learning. An excellent mathematics program requires effective teaching that engages students in meaningful learning through individual and collaborative experiences that promote their ability to make sense of mathematical ideas and reason mathematically.

Access and Equity. An excellent mathematics program requires that all students have access to a high-quality mathematics curriculum, effective teaching and learning, high expectations, and the support and resources needed to maximize their learning potential.

Curriculum. An excellent mathematics program includes curriculum that develops important mathematics along coherent learning progressions and develops connections among areas of mathematical study and between mathematics and the real world.

Tools and Technology. An excellent mathematics program integrates the use of mathematical tools and technology as essential resources to help students learn and make sense of mathematical ideas, reason mathematically, and communicate their mathematical thinking.

Assessment. An excellent mathematics program ensures that assessment is an integral part of instruction, provides evidence of proficiency with important mathematics content and practices, includes a variety of strategies and data sources, and informs feedback to students, instructional decisions, and program improvement.

Professionalism. In an excellent mathematics program, educators hold themselves and their colleagues accountable for the mathematical success of every student and for their personal and collective professional growth toward effective teaching and learning of mathematics.

Source: *Principles to Actions* (NCTM 2014, p. 5)

makers, school and district administrators, and teachers, coaches, and specialists of mathematics that are based on the NCTM's core principles. NCTM defined a set of principles in 2000 that described "features of high-quality mathematics education" (NCTM 2000, p. 11). The updated principles (see **sidebar**, above left) reflect more than a decade of experience and new research evidence about excellent mathematics programs. Within the new publication, each of the six principles—Teaching and Learning, Access and Equity, Cur-

Mathematics Teaching Practices

Establish mathematics goals to focus learning. Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.

Implement tasks that promote reasoning and problem solving. Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.

Use and connect mathematical representations. Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving.

Facilitate meaningful mathematical discourse. Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments.

Pose purposeful questions. Effective teaching of mathematics uses purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships.

Build procedural fluency from conceptual understanding. Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.

Support productive struggle in learning mathematics. Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.

Elicit and use evidence of student thinking. Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning.

Source: *Principles to Actions* (NCTM 2014, p. 10)

riculum, Tools and Technology, Assessment, and Professionalism—are further defined, examined for unproductive and productive beliefs, linked to effective practices, and illustrated with examples.

Three elements of *Principles to Actions* should be highlighted. First, in a chapter focusing on teaching and learning, the publication describes and illustrates eight Mathematics Teaching Practices (see **sidebar**, above right), which need to be consistent and essential components of every mathematics lesson. Second, each of the six principles includes discussion of unproductive and productive beliefs as part of an honest appraisal of obstacles to implementation as well as suggestions for overcoming these obstacles. Third, *Principles to Actions* presents three sets of actions—one set for leaders and policy-makers in districts and states or provinces, including commissioners, superintendents, and other

High School Mathematics

Efforts to achieve curricular coherence in mathematics at the high school level are particularly challenging, given the typical sequence of courses and topics, in which the study of geometry is often isolated as a separate course and statistics is grafted onto courses in stand-alone units rather than naturally connected to related topics (e.g., using a visual line of best fit to lead into a formal study of linear functions). Some schools have successfully reconfigured their programs as integrated sequences of courses that address algebra, geometry, statistics, probability, and discrete mathematics topics across all grade levels, allowing students to revisit these topics at increasingly sophisticated levels and make connections among them. Such reconfiguration requires developing mathematical reasoning and helping students see how, for example, a probability problem can be solved by use of a geometric model or how geometric transformations of shapes can be performed through the use of matrices in algebra.

All high schools should reevaluate their mathematics programs to determine whether the current sequence of courses is preparing students for the demands of a workplace that will require more than the mastery of isolated mathematics skills. Such a reevaluation might require that teachers know “how and why mathematical models are derived,” how to “create their own models,” and how to “think about the relationship between the models and the mathematics that is integrated” (Keck and Lott 2003, p. 131). Efforts to build coherence across the high school curriculum are of paramount importance.

Source: *Principles to Actions* (NCTM 2014, p. 75)

central office administrators; one for principals, coaches, specialists, and other school leaders; and one for teachers.

THE SECONDARY SCHOOL MATHEMATICS CURRICULUM

A section of *Principles to Actions* describes the impact of a well-designed curriculum. The authors discuss the difference between a *standard*, the *ends* that can be thought of as the product or goal of a mathematics program, versus the *curriculum*, the *means* that are used to accomplish those goals (e.g., textbooks, instructional materials, activities, and lesson plans). In the publication, teachers and administrators of high schools are encouraged to rethink the way that the mathematics curriculum is designed. In far too many cases, high school courses are disjointed and do not help students make connections across content areas. This issue is addressed in *Principles to Actions*, as illustrated in the **sidebar** (above).

TAKING ACTION

Finally, *Principles to Actions* argues that ensuring mathematical success for all students will take teachers who effectively plan and use the core set of eight mathematics teaching practices as a framework for strengthening the teaching and

learning of mathematics. Mathematical success for all students will take principals, coaches, specialists, and other school leaders who are committed to supporting teachers in their efforts to engage students in important mathematics and who fully understand and assist teachers in consistently planning for and implementing the core mathematics teaching practices. High levels of student success in mathematics also require that leaders and policymakers in districts and states or provinces, including commissioners, superintendents, and other central office administrators, take action to ensure that all teachers have the resources and supports that are essential to the enactment of effective mathematics teaching. A set of specific recommended actions is provided for each group of stakeholders to meet these expectations. Reading, discussing, strategizing about, and acting on these actions is the next step for all of us who care deeply about significantly raising student achievement in mathematics.

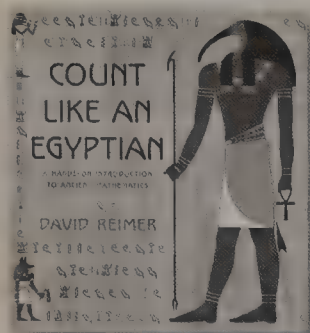
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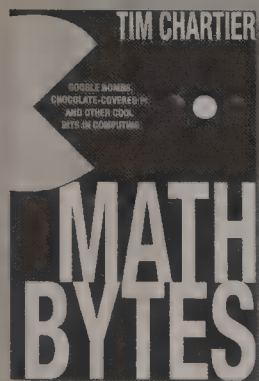
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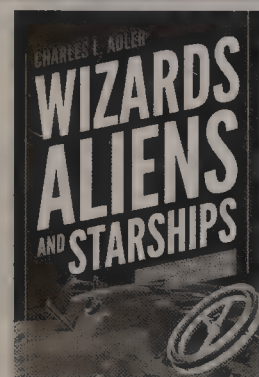
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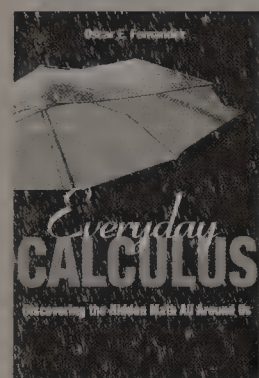
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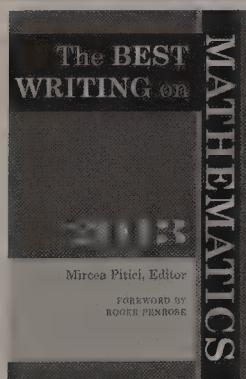
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TECHNOLOGY-ENHANCED DISCOVERY

Tracking the foci of conic sections reveals

Exploration, innovation, proof: For students, teachers, and others who are curious, keeping your mind open and ready to investigate unusual or unexpected properties will always lead to learning something new. Technology can further this process, allowing various behaviors to be analyzed that were previously memorized or poorly understood.

This article shares the adventure of one such discovery of exploration, innovation, and proof that was uncovered when a teacher tried to find a smoother way to model conic sections using dynamic technology. When an unexpected pattern regarding the locus of an ellipse's or hyperbola's foci emerged, he pitched the problem to a ninth grader as a challenge, resulting in a marvelous adventure for both teacher and student. Beginning with the evolution of the ideas that led to the discovery of the focal locus and ending with the significant student-written proof and conclusion, we hope to inspire further classroom use of technology to enhance student learning and discovery.

TEACHER-POSED PROBLEMS

Hidden Parabolic Behavior

Studying the graph of a generic quadratic function, $f(x) = ax^2 + c$, most students eventually learn that the value of a determines the “width” (or, more precisely, the vertical stretch) of the parabola and that c gives its vertical translation. When exploring these curves, I ask my students to create graphs on their calculators with parameters that are defined by sliders (see **fig. 1**).

By dragging the sliders for a and c , the parabola changes in real time while students deduce the parameters' effects on the graph. One particularly nice result of this approach is that students see all these quadratics as part of a single function family. Each individual graph is one part of a continuum. This realization also allowed me to discuss precisely what happens when $a = 0$. Although some might argue that $a = 0$ does not define a quadratic, I see it as a degenerate parabola. As a student, I learned that degenerate cases occur when equations fail. With technology, I now understand them

ANCED ERY

surprising locus.

Chris Harrow and Lillian Chin

as the boundary curves between subgroups of an equation family. In this case, the $a = 0$ degenerate parabola marks the boundary between upward and downward-facing quadratic graphs.

During a USACAS conference presentation roughly ten years ago, a speaker acknowledged these two transformations before asking what happened when b varies in $y = ax^2 + bx + c$. Adding a slider for b and enabling a geometry trace to the parabola's vertex produced a surprise (see **fig. 2**).

Every year, I assign the following project to algebra or precalculus students. The mathematics is not particularly difficult, but I assign groups of two or three because I want my students to test the strength of their reasoning before submitting solutions. The project is as follows:

The expanded polynomial form of a quadratic function is $y = Ax^2 + Bx + C$.

1. What would an eighth grader say happens to the graph of a quadratic when the value of A changes?



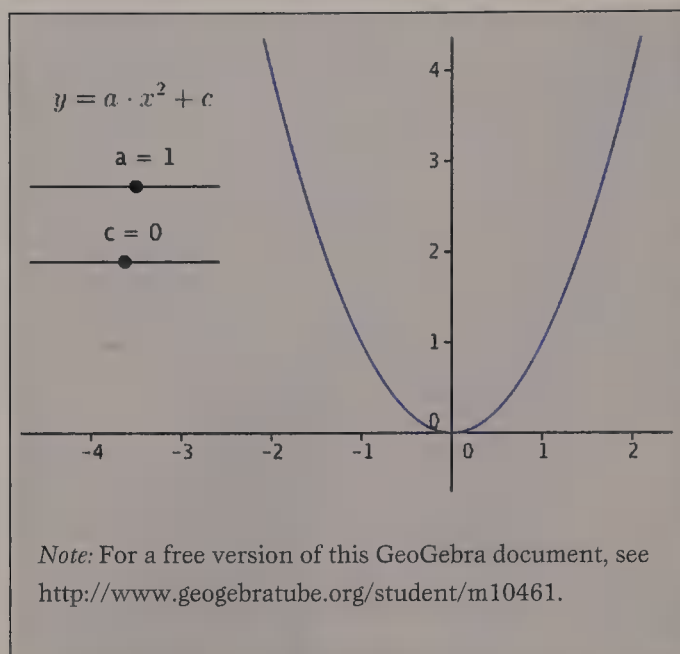


Fig. 1 A class activity investigates how changing a and c transforms $f(x) = ax^2 + c$.

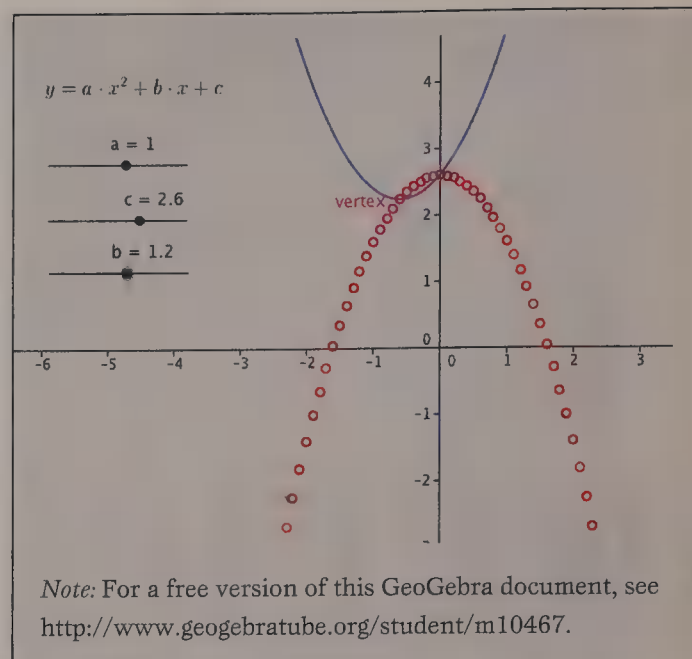
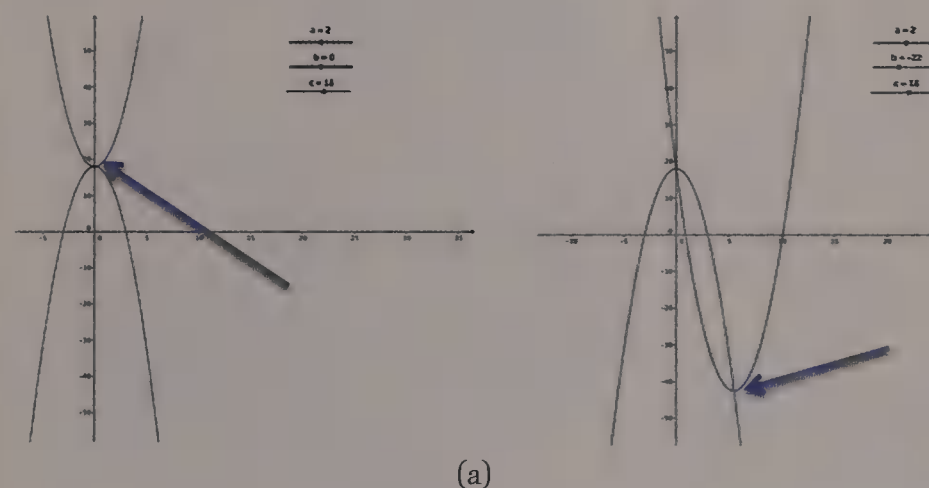


Fig. 2 The trace of the vertex of $y = ax^2 + bx + c$ when b varies is marked in red.

As we changed b , we began to realize that the vertex of the parabola was following a curve. We discovered the curve to be a parabola with equation $y = -ax^2 + c$.



(a)

In order to algebraically prove our hypothesis, we set the equation of the standard parabola ($y = ax^2 + bx + c$) equal to the equation of the parabola that we believed the standard parabola's vertices would always reside on ($y = -ax^2 + c$). We substituted the x -value of the vertex of the standard parabola ($-b/(2a)$) in for x and then simplified:

$$\begin{aligned} -a \cdot \frac{b^2}{4a^2} + c &= \frac{ab^2}{4a^2} + \frac{-b^2}{2a} + c \\ \frac{-b^2}{4a} + c &= \frac{b^2}{4a} + \frac{-2b^2}{4a} + c \\ \frac{-b^2}{4a} + c &= \frac{-b^2}{4a} + c \end{aligned}$$

(b)

Fig. 3 Students claim that $y = -ax^2 + c$ is the locus of the vertex of $y = ax^2 + bx + c$ when b varies (a). Students prove that $y = -ax^2 + c$ is the locus of the vertex of $y = ax^2 + bx + c$ when b varies (b).

2. What would an eighth grader say happens to the graph of a quadratic when the value of C changes?
3. You already knew the answers to questions 1 and 2, but what happens to the graph of a quadratic when the value of B changes? Be specific, and prove your claim.

Figures 3a and 3b show the discovery and proof stages from two of my students who tackled the project during the fall 2012 semester. Having achieved a tautology using reversible algebraic manipulations, the students knew that they had proven their claim.

Although this problem and its student answer are nice, I was surprised that I had never seen the transformational effects of b before. How could I have worked with quadratics for so long without seeing this? The lovely way this example demonstrated quadratic behavior encouraged me to explore other conic sections using the same approach.

Parameters to Investigate

One of my first investigations into conics with sliders was manipulating a completed square form:

$$\frac{(x-h)^2}{a} + \frac{(y-k)^2}{b} = 1$$

The denominators are intentionally unsquared to allow the document (see **fig. 4**) to produce both ellipses and hyperbolas, depending on the relationship between a and b .

I use this document on the first day of the conics unit. Students experimentally discover the effects of h , k , a , and b on the graph and eventually explain the parameter differences between the algebraic

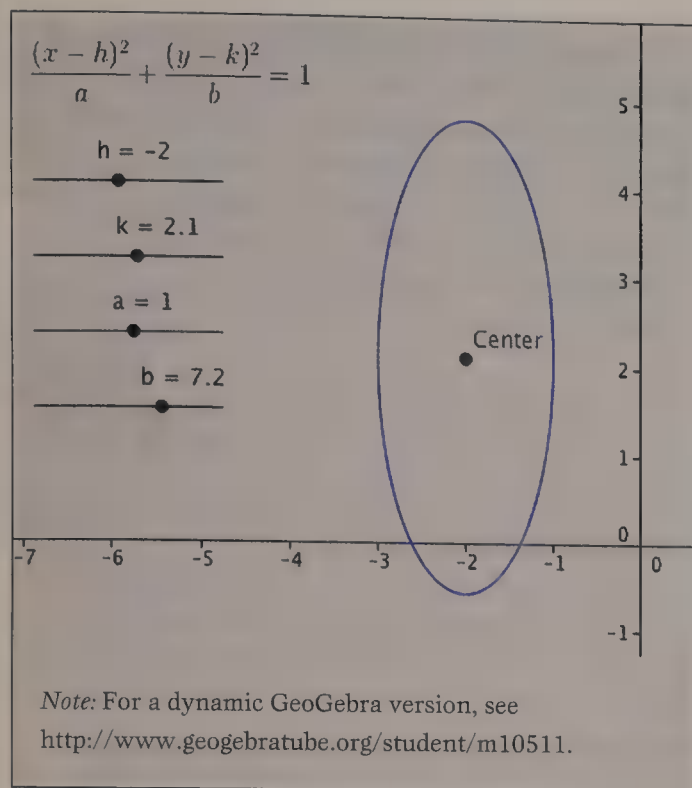


Fig. 4 The equation with $0 < a < b$ produces an ellipse with a vertical major axis.

forms of ellipses and hyperbolas, the algebraic differences between horizontal and vertical conics, and how the values of h and k determine a conic's center. The usefulness of this form helps motivate students to learn algebraic techniques in later lessons to transform other equations into this more "readable" form.

Physical models of the cone usually show how all four conic sections can be extracted from a single object, but the demonstration in **figure 4** could not produce parabolas. Not satisfied, I created another document using the polynomial form of all conics, $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. (A GeoGebra version excluding the Bxy rotation term is at <http://www.geogebra tube.org/student/m10660>.)

Next, I produced the entire conic family from a single algebraic source while also tracking the centers of the conic sections. Inspired by my success, I added generic coordinates for the conic foci so that users could see the locations of the foci change dynamically as they manipulated the coefficients in the equation of the conic polynomial (<http://www.geogebra tube.org/student/m12946>). Just as the effects of b in $y = ax^2 + bx + c$ were unexpected, **figure 5** shows the surprising result when the value of E changes in horizontal conics.

When the conic is a horizontal hyperbola, varying E moves its foci along an apparently elliptical path. Likewise, the foci of a horizontal ellipse move along a hyperbolic path when E changes. The same behavior occurs for vertical conics when D changes.

I was astonished again. Here was a glorious problem that revealed itself only when technology was used to process far more cases nearly instan-

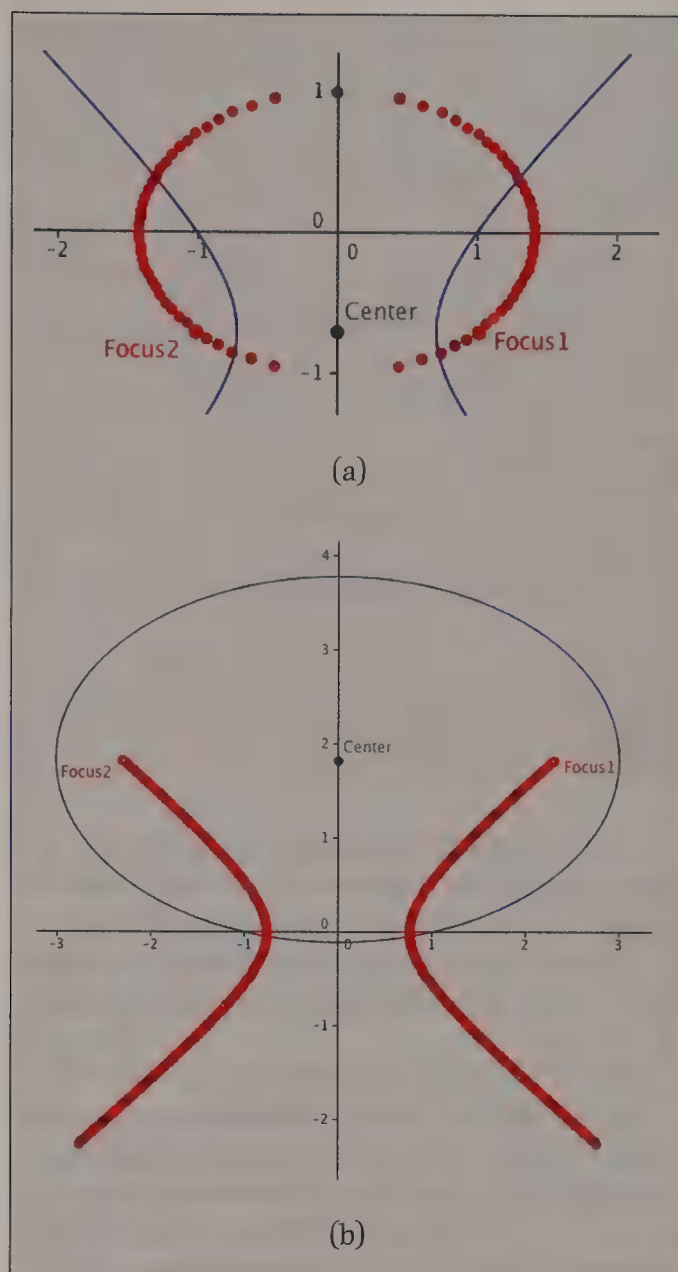


Fig. 5 The foci of a horizontal hyperbola are plotted in red (a). The foci of horizontal ellipses created using sliders can be tracked (in red) to generate a hyperbola (b).

taneously than could ever be done by hand. After I showed the problem to my students, Lilly, one of the ninth graders, jumped at the possibility of exploring this focal behavior. Eventually, she independently and enthusiastically proved this embedded property. Lilly wrote the next section of this article.

STUDENT-WRITTEN PROOF

Hidden Conic Behavior

I was used to being astounded by my teacher. A large focus of his class was on transformations, and he used the aforementioned class activities to help us gain a better understanding. One of the most noteworthy moments in that class was when we came to the informal realization that hyperbolas could be seen as ellipses "passing through infinity," wrapping around the coordinate plane. Even though I understood the more formal derivations of intersecting a double-napped cone with a plane, the idea still stuck with me.

EXTENSIONS AND APPLICATIONS

All the resources used in this article are open source and freely available. Here are some more ways that you can use technology in the classroom to motivate your students.

- This blog post (<http://bit.ly/VBwZaR>) shows the parabola problem from the beginning of this article as posed in class. This post (<http://bit.ly/PiWHNJ>) shows an extension not yet assigned.
- For teachers interested in showing the conic section proof to their class, there are several challenging investigative opportunities for interested students. Although we did not have time to fully investigate these avenues, advanced students may be interested in the following:

1. Explore degenerate conic cases as the boundaries between different conic forms.
2. Add a B slider for the term Bxy to play with rotated conics.
3. Perform this locus operation successively and look for relationships in the sequence of conic sections generated.

So when he first showed this property to me, I was not surprised. Another example of the interconnectedness of all conic sections through transformations? Go figure. However, I was not simply content with just assuming this property was true; I had to know why.

I first delved into the literature to search for any mention of this relationship. Although this property was new to us, conic sections have been studied for over 2000 years, so I was sure I would find some mention of this relationship. However, I was unable to find any direct or indirect acknowledgement of this property, and the mathematicians to whom we spoke seemed just as surprised about the relationship.

Inspired by the incredible beauty of the relationship, I embarked on the following proof. I focused only on varying E for the horizontal orientation of a conic section because all other orientations could be obtained through rotations or reflections. These transformations would not affect the general shape of the foci's locus. For example, an analogous proof exists for vertical conic sections if one varies D rather than E . Without loss of generality, I can focus solely on horizontal conic sections.

Claim

The locus of a horizontal ellipse's foci when varying E in the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ will result in a hyperbola. The locus of a horizontal hyperbola's foci will result in an ellipse.

Proof

Start with the general equation of a conic section, $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where A, B, C, D, E are real numbers and F is a real number chosen so that the equation has real solutions (x, y) .

As previously stated, we will focus on horizontal conic sections where $B = 0$, leaving $Ax^2 + Cy^2 + Dx + Ey + F = 0$. This equation could represent either an ellipse or a hyperbola, depending on the signs of A and C . If A and C are the same sign, then we have an ellipse. Otherwise, we have a hyperbola.

To find the foci, manipulate this form by completing the square:

$$A \left(x^2 + \frac{D}{A}x + \left(\frac{D}{2A} \right)^2 \right) + C \left(y^2 + \frac{E}{C}y + \left(\frac{E}{2C} \right)^2 \right) = -F + \frac{D^2}{4A^2} \cdot A + \frac{E^2}{4C^2} \cdot C$$

$$\text{Let } S = -F + \frac{D^2}{4A^2} \cdot A$$

and divide both sides of the equation by AC to clean up:

$$\frac{\left(x + \frac{D}{2A} \right)^2}{C} + \frac{\left(y + \frac{E}{2C} \right)^2}{A} = \frac{S + \frac{E^2}{4C^2} \cdot C}{AC}$$

Therefore, the center of the conic section is

$$(x_c, y_c) = \left(-\frac{D}{2A}, -\frac{E}{2C} \right).$$

Rewriting the equation, we have

$$\frac{(x - x_c)^2}{C} + \frac{(y - y_c)^2}{A} = \frac{S + Cy_c^2}{AC}$$

$$\frac{(x - x_c)^2}{\frac{S + Cy_c^2}{A}} + \frac{(y - y_c)^2}{\frac{S + Cy_c^2}{C}} = 1. \quad (1)$$

To find the foci (x_f, y_f) of the horizontal conic sections, we consider the two shapes separately.

For an ellipse: Let a be the length of the semi-major axis and b be the length of the semiminor axis. In an ellipse, the foci are along the major axis and are $\sqrt{a^2 - b^2}$ away from the center.

From (1),

$$a^2 = \frac{S + Cy_c^2}{A} \text{ and } b^2 = \frac{S + Cy_c^2}{C},$$

making the distance between each focus and the center

$$\sqrt{\frac{S + Cy_c^2}{A} - \frac{S + Cy_c^2}{C}}. \quad (2)$$

For a hyperbola: The foci are still on the major (transverse) axis but are now $\sqrt{a^2 - b^2}$ away from the center. So

$$a^2 = \frac{S + Cy_c^2}{A} \text{ and } b^2 = -\frac{S + Cy_c^2}{C},$$

and the focus-center distance is

$$\sqrt{\frac{S + Cy_c^2}{A} - \frac{S + Cy_c^2}{C}}.$$

This is the same expression as the elliptical case. This means the same equation for the locus of the foci holds for both curves. Whether equation (1) gives an ellipse or a hyperbola, equation (2) locates the foci on the horizontal axis in relation to the center:

$$(x_f, y_f) = \left(x_c \pm \sqrt{\frac{S + Cy_c^2}{A} - \frac{S + Cy_c^2}{C}}, y_c \right)$$

Because the conic section is horizontal, we can just substitute y_f for y_c . This substitution is not just switching one constant with another; we are replacing the E -dependent constant y_c with the variable y_f which is dependent only on x_f . This switch gives us an equation defined by x_f and y_f rather than by E .

Substitute and manipulate to get to standard form:

$$\begin{aligned} x_f &= x_c \pm \sqrt{(S + Cy_f^2) \cdot \frac{C - A}{AC}} \\ (x_f - x_c)^2 - \frac{C(C - A)}{AC} y_f^2 &= \frac{S(C - A)}{AC} \\ \frac{(x_f - x_c)^2}{(C - A)S} - \frac{y_f^2}{\frac{S}{C}} &= 1 \end{aligned} \quad (3)$$

This general equation gives the locus of the foci for both horizontal ellipses and hyperbolas.

What remains to be proved is that equation (3) represents an ellipse if the original equation was a hyperbola and that equation (3) represents a hyperbola if the original equation was an ellipse.

These properties shall be shown by analyzing the denominators of the two main fractions. Since S/C is present in both denominators, its sign does not matter. The proof focuses only on the sign of $(C - A)/A$. If (1) is a horizontal ellipse, then A and C have the same sign and $|C| > |A|$. Thus, $C - A$, C , and A all have the same sign, making $(C - A)/A$ positive. Therefore, both denominators will have the same sign of $+S/C$, making (3) a hyperbola.

If (1) is a hyperbola, A and C have different signs. If $C < 0$, then $A > 0$, and $C - A < 0$. Likewise,

if $C > 0$, then $A < 0$ and $C - A > 0$. So, $C - A$ must have the same sign as C . Thus, $(C - A)/A$ is negative, and both denominators have the same sign of $-S/C$, making (3) an ellipse. QED.

OUR REFLECTIONS

This journey—exploration, pattern finding, proof—has truly been inspirational to me. With nothing but logic and clear mathematical understanding, I have proved a universal truth! Unlike the problems given in homework or the two-column proofs from geometry, I was not just proving the self-evident or rewriting tautologies. Instead, I was going on my own mathematical adventure, stumbling on a beautiful property that caused me to wonder, “It seems to be true in this case, but what about all cases?”

This whole idea of discovery is what makes me love the revelatory nature of algebraic and graphical technology in the classroom. Without GeoGebra computing literally thousands of cases per slider tick, I would not have been able to notice the embedded conic section behavior or marvel at the intertwined unity of the conic sections. For me, technology is a tool, not a crutch, allowing me to better explore unknown patterns and motivate algebraic proof.

I hope that technology will enhance your classroom just as it has enhanced my learning. I love every minute I spend playing with proofs and ideas and continue to creatively pursue mathematical research.



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MULTIPLE REPRESENTATIONS SIERPINSKI

A task using multiple representations helps

J. Vince Kirwan and

To understand multiple representations in algebra, students must be able to describe relationships through a variety of formats, such as graphs, tables, pictures, and equations. NCTM indicates that varied representations are “essential elements in supporting students’ understanding of mathematical concepts and relationships” (NCTM 2000, p. 67). By using assorted representations, students have the opportunity to make sense of each display and see how each relates to the others (Yerushalmy and Schwartz 1993). For example, a table organizes numerical values for variables that satisfy an equa-

tion, whereas the equation describes the relationship between these variables. No single representation is superior to another because each highlights different mathematical attributes of the relationships in context (Friedlander and Tabach 2001).

Difficulties that students have with learning algebra are often rooted in a lack of understanding associated with various representations (Kieran 2007), especially symbolic notations (Poon and Leung 2010). Having students work with symbolic representations—that is, algebraic expressions or equations—is a common classroom approach to teaching algebra. Further, we teachers often



AND CONNECTIONS WITH THE TRIANGLE

Students write explicit algebraic equations.

Jennifer M. Tobias

incorporate varied representations when teaching a topic, such as graphing an equation in two variables. However, if students are not making sense of, understanding, or connecting these representations, they will often struggle with representing the situation symbolically.

THE SIERPINSKI TRIANGLE INSPIRES INVESTIGATION

NCTM's Algebra Standard expects students in grades 9–12 to “. . . select, convert flexibly among, and use various representations” (2000, p. 296). Thus, this activity was designed to engage students

with using varied representations as they worked within a unit on exponential relationships. The goal was to have students develop connections among representations and then generalize the relationship symbolically through this connecting process.

Students in this class were given the first four iterations of the Sierpinski triangle fractal as shown in **figure 1** and asked to investigate the relationship between the step and the number of black triangles displayed in each step.

Before reading further, take a moment to determine the pattern regarding the progression of the

number of black triangles. If someone asked you to describe and explain this relationship, how would you do so?

When examining the number of black triangles, students were asked to represent their findings in pictorial, tabular, graphical, and symbolic formats. **Figure 1** served as the basis on which students developed their representations. Students were not told how this figure was generated because one goal was for students to develop a pictorial representation based on their interpretations of the patterns within the figure. Students first worked individually or in small groups, developing these representations in the order they preferred. Afterward, students participated in a whole-class discussion. This discussion allowed the students to share and compare their representations along with the understandings and connections they developed. During this time, each representation was overtly

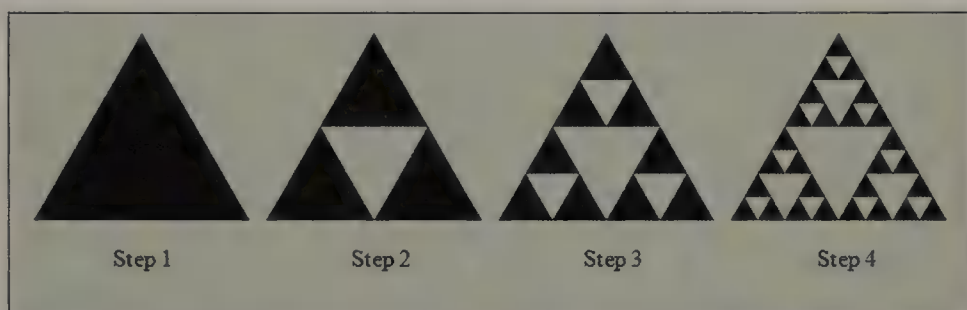


Fig. 1 A pictorial representation was the basis for investigation.

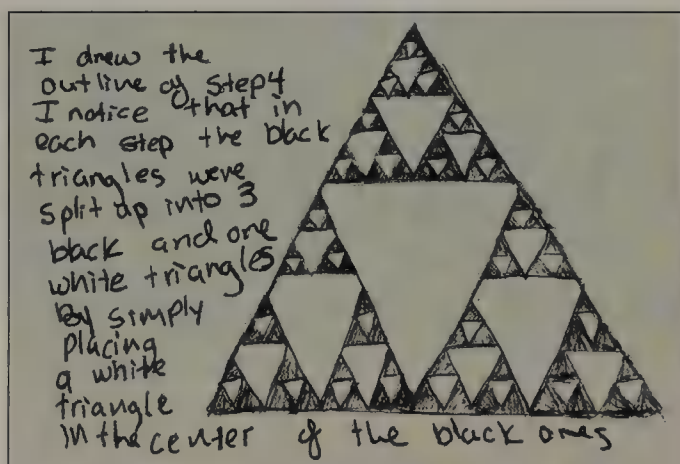


Fig. 2 One student described the process of generating the fractal.

steps	Black Δ s	
+1 (1	1	As the steps ↑, the black Δ ↑ by a multiples of 3.
+1 (2	3	
+1 (3	9	
+1 (4	27	
+1 (5	81	
+1 (6	243	

Fig. 3 Another student attended to covarying quantities in a table.

discussed. The representations section below follows the same outline as the class discussion and is illustrated throughout by various students' work. This is followed by teaching strategies to help students make connections among multiple representations.

MULTIPLE REPRESENTATIONS INVITE COMPARISON

Pictorial Representations

While investigating the relationship pictorially, students determined how each step in the Sierpinski fractal was generated. Some recognized a recursive relationship within **figure 1**. One student described this relationship as partitioning each black triangle in step 4 by placing a white inverted triangle at the center, creating three smaller but similar black triangles and step 5 of the Sierpinski triangle (see **fig. 2**).

Another student summarized this pattern, writing that "each black area breaks into 3 . . . so you multiply the previous number [of black triangles] by 3." Both students appeared to be transitioning from a purely pictorial standpoint toward a position where they could begin quantifying the relationship that they were investigating.

Although cutting each black triangle with a white triangle was a prevalent strategy, a second iterative technique was also discovered. During the process of sharing their work with the class, several students noticed that replicating three copies of a step and arranging them in a pyramid generated the subsequent step. The students using this iterative process seemed to draw on the self-similarity aspect of the Sierpinski triangle fractal in generating each step. Note that the students' recursive reasoning in this replication view also explains the tripling of the solid triangles in each step.

Focusing on not only the created object but also the creation process is important for each successive step in the Sierpinski fractal. This was one difficulty that students encountered: They wanted to focus on the created product rather than the process of creation or the connections between their representations. The self-similarity aspect of fractals highlights both the product of a previous step and its use through a process to create the subsequent step. Because representation "refers both to process and to product" (NCTM 2000, p. 67), students need to attend to both their answer and how they developed their answer. This creation process is a critical component for students in making sense of how a pattern evolves. Attending to how one pictorial representation is changed into another can help yield insights in the same way that studying the change in values in a table helps determine the type of functional relationship.

Accuracy is important. Having students discuss conjectures and explain their thinking brought misconceptions to the forefront.

Tabular Representations

As students worked with tabular representations, they coordinated two quantities—the step and the number of black triangles. Some students indicated that as the steps increased, the number of black triangles increased by a factor of 3.

The student whose work is shown in **figure 3** found that step 6 would have 243 black triangles. Another student quantified this relationship a little more specifically, stating that as “each step . . . is increased by 1, the previous number of triangles is multiplied by 3.” Students working in a systematic fashion with tables were able to extend the relationship beyond the initial four steps.

However, not every student attended to the covarying quantities in the tables. Some focused solely on the number of black triangles being cut into three. Changes in both the step and the number of black triangles together describe the pattern in a holistic fashion and also prompt students to transition from thinking recursively to coordinating the covarying quantities and ultimately to graphing and writing equations for two-variable relationships.

Graphical Representations

As students worked graphically, they noted that increasing each step by 1 would result in the number of black triangles being increased by a factor of 3. One student used the tripling pattern to rationalize the increasing steepness and the gap between points in the graphical representation (see **fig. 4**). As students worked with their graphs and other representations, some tended to attribute specific properties incorrectly. These students indicated that because the graph was curved upward, it must be parabolic (see **fig. 5**). One student even indicated that “half a parabola” and “exponential growth” were equivalent ideas. Although connection to other topics is desirable, accuracy is important. Having students discuss conjectures and explain their thinking brought misconceptions to the forefront and helped students address misunderstandings.

Although few students encountered difficulty in creating tabular or graphical displays, the subtle differences regarding the domain of the relationship often eluded them. For example, when students converted from tables to graphs, they often connected the points on the graph (see **fig. 5**). The represented relationships are similar but no longer equivalent because of changes in the domain. For example, although the ordered pairs (1, 1) and (2, 3) appear

in the table and graph, the graph also includes additional ordered pairs, such as $(1.5, \sqrt{3})$, that the table does not. The graph illustrates a continuous relationship between the quantities, but the table indicates a discrete relationship. This subtle yet critical difference is important to address in preparing students to study other noncontinuous relationships, such as rational functions with removable discontinuities. Problem context can help present a platform for addressing the misconception that the representations are equivalent.

Symbolic Representations

After discussing pictorial, tabular, graphical, and verbal representations, students developed symbolic generalizations of the number of black triangles per step in the Sierpinski triangle fractal. This representation

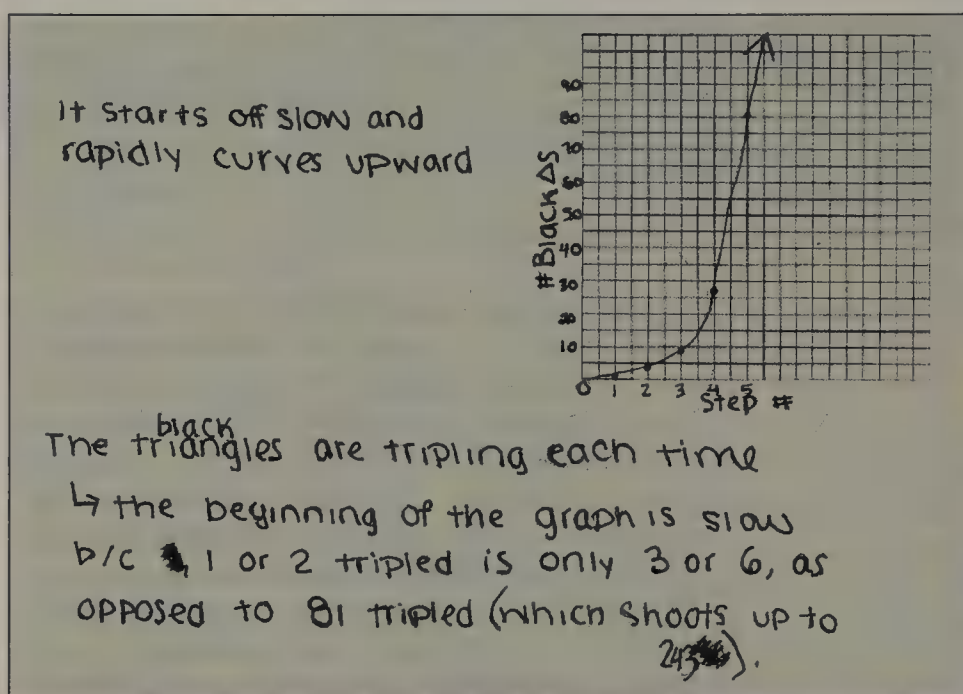


Fig. 4 A student connected the generative process and the graph.

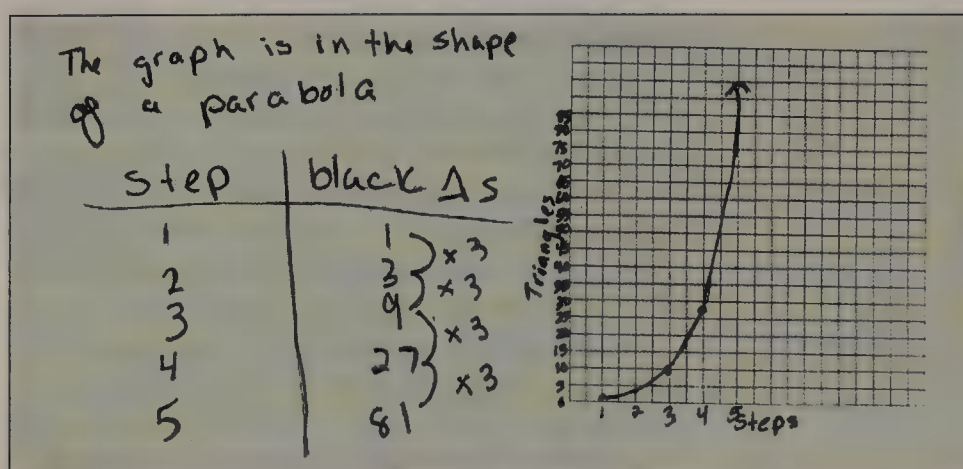


Fig. 5 This student's work showed a misconception about shape.

$B = 3^{(x-1)}$

I knew there had to be a 3 included in the equation b/c of the y values ↑ by multiples of 3 + I knew it had to deal w/ exponents b/c it was using multiplication for black As.

Fig. 6 One student rationalized from the tabular representation.

$y = 3^{x-1}$ ← y = # of black As
x = step #

The curve of the graph looked like an exponential curve, which influenced me to produce an exponential equation. The 3 in my equation resulted from the tripling pattern I noticed.

Fig. 7 Another student rationalized from the graphical and pictorial representations.

Step #	Black Triangles
1	1
2	3
3	9
4	27
5	81

Being added by one

Being multiplied by three

Fig. 8 This student decomposed numeric values in the table.

tended to be the most difficult for students. Many did not know how to generalize the relationship of black triangles in each step or what numbers to use. Students commented that they did not know where to start or were not sure what to do. To help overcome this obstacle, we directed students to focus on the process of creating the representation rather than on the created object. For students who did not automatically make connections, we also asked whether any of the previous representations they had generated could help them and how. Because each student may think about the relationship differently, multiple representations provided more than one way to rationalize the symbolic representation.

Students who were successful in finding a symbolic expression or equation typically made connections between their equation and other representations. One student was able to explain why the growth factor in the number of solid triangles is three by referring to how the number of black triangles changes in each step (see fig. 6).

Other students developed their equation by referring to the graph and pictures that they drew (see fig. 7).

Finally, some students created equations from the patterns that they found within their tables.

One student decomposed and rewrote values from her table in an exponential form (see fig. 8) and then developed an equation.

Related to this exponential form, another student argued for the exponent, writing, "All of the amounts [numbers] of black [triangles] are products of 3" but they are a step off, so I subtracted 1 from the n to correct that."

As students worked with this task, they investigated a relationship using pictorial, tabular, graphical, and symbolic representations. They considered the relationship from a variety of viewpoints and looked to make sense of their creations by linking them with other representations. Using multiple representations seemed to help them think about the problem in additional ways besides algorithmically. Making their own connections between representations helped students develop a symbolic representation.

TEACHING STRATEGIES ENCOURAGE MAKING CONNECTIONS

Having students explicitly verbalize what they know was a powerful strategy in helping determine what they actually understood. For example, we asked students who misidentified their curved graph as parabolic to explain what a parabola was and how they could recognize it in other representations; we also asked them to consider other curved graphs superimposed over their exponential graph. Typically, just asking the question "What is a parabola?" prompted students to reconsider their statement. In addition, this verbalization strategy served as a technique for the teacher to help identify student misconceptions.

Having students work in small groups and present their work to the class was also beneficial. In these settings, students were able to communicate their ideas and understanding to one another. For example, when students worked with pictorial representations, they often wanted to fixate on the picture they created. To help shift their focus, we asked them to share and explain their work to their peers. In doing so, they had to explain how they went about generating each representation. During this time, other students or the teacher would ask the presenting students why they proceeded in one way or how they knew whether they were correct. Students sometimes made connections during this explanation process because they had to make their thinking explicit. Sharing what they had done and why they had done so enabled them to consider not only what they knew but how they knew it.

Another useful strategy was to avoid dictating the order in which students had to develop each representation. Students could create representations in the order that appeared most reasonable to them. Some first developed tables of values, while others developed pictorial representations as their initial

step. This option also seemed to help students develop connections between representations by working with what appeared natural to each student.

As students build connections between representations and make sense of the relationship, patience is a key element. Creating new connections between ideas is cognitively difficult. Students will most likely need time to create these connections (as well as reminders to focus on the process and not just the product). Considering other patterns and relationships gives students a chance to apply their current understandings to new situations, allowing them to refine their conceptions and processes.

This Sierpinski triangle task can easily be extended. For example, students could consider other relationships within the fractal, such as the proportion of area covered by these solid triangles or the perimeter of the black triangles (see Brown 2005). In addition, students could consider analogues to these problems (and then some!) from the Sierpinski pyramid fractal. The iterative construction process could also be extended to other shapes, such as squares, posing the same sorts of investigations. Further, many of these could be explored dynamically from various online resources, such as NCTM's Illuminations. These extensions offer students additional relationships to explore while generalizing connections among representations.

UNDERSTANDING MADE EXPLICIT REFINES REASONING

Some students may initially notice connections between pictures and tables; others may first connect tables and symbolic representations. Making explicit what they notice about these representations and what that means in terms of the pattern they are studying can benefit the presenting students—as well as those listening—in connecting to other representations. By making their understanding explicit, students are able to refine what they notice to be the essential components in creating connections. These reasoned links can then provide additional access points for students to draw on or connect to.

As students spend more time working with representing other relationships, they may notice general themes beginning to emerge. Allowing time for students to consider these overarching ideas helps them situate the ideas not only within specific problem contexts but also within algebra and mathematics as a whole. This situating of ideas within their knowledge scheme then provides students with a power-

By making their understanding explicit, students are able to refine what they notice to be the essential components in creating connections.

ful skill when working with mathematics.

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Editor's note: For photographs of a three-dimensional sculpture based on a Sierpinski triangle, see "Mathematical Lens," *MT* April 2014, pages 414 and 415.

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Authenticity of thematical Modeling

The choice and context of authentic problems—such as designing a staircase or a soda can—illustrate the modeling process in several stages.

Dung Tran and Barbara J. Dougherty

Some students leave high school never quite sure of the relevancy of the mathematics they have learned. They fail to see links between school mathematics and the mathematics of everyday life that requires thoughtful decision making and often complex problem solving. Is it possible to bridge the gap between school mathematics and the mathematics in students' lives while developing a positive attitude toward mathematics? Incorporating tasks that are situated in real-life contexts and involve modeling can address these demands. These types of tasks may help students see the purpose of mathematics.

Consider the two versions of the Bus problem (see **fig. 1**). Compare the expectations for

student responses in these two versions. What do you notice?

The mathematical content needed to solve both problems is identical, but the requirements for student responses are different. Which version do you think produces better student responses? When the first problem was presented to students in grades 6–9, the correct answer of 7.5 buses was given by 91% of the 124 students. However, the second version given to fifth graders was answered correctly by 95% of 161 students. The comparison of student achievement is warranted.

To help us think about these differences, let's examine criteria we can use to choose tasks that embody modeling. Our choices may offer some ways to create better student performance. We provide examples to illustrate

Two Versions of The Bus Problem

Version 1: Three hundred sixty students go by bus on a school trip. Each bus can hold 48 students. How many buses are required?

Version 2: All students in the school will go on a school trip together on the 15th of May. You and the other organizing students have decided that everyone will go by bus and that you will order the buses. You have seen in the student rosters that there are 360 students in the school. Your teacher said that you can order the buses from Swebus, and that each bus can hold 48 students. Fill in the order form, which you are going to send to Swebus to order the buses. (Palm 2008, p. 43).

Swebus – Bus order

Your name:.....

School:.....

Date of the trip:.....

Number of buses to order:.....

Other requirements:.....

modeling activities linked to the Common Core State Standards for Mathematics (CCSSM) (CCSSI 2010) and present two frameworks—authenticity and modeling hierarchy—to analyze problems. We use these frameworks in selecting and designing tasks to promote student engagement and learning.

AUTHENTICITY

Given the similarity of the two versions of the Bus problem, it seems unusual that there would be a difference in student performance. However, a task that is made more authentic by completing an actual order form may trigger student engagement leading to a solution that is more appropriate.

What makes a task authentic? Palm (2008) suggested that an authentic task is one in which the situation described includes a question or situation that has occurred or that might occur in real life. The task is realistically described, and the conditions under which the solution process takes place are a reasonable simulation of the real situation (see **fig. 2** for details).

In addition to authenticity, we must also consider the stages for solving problems that involve modeling. A modeling cycle includes identifying variables that are essential to formulate a mathematical problem that represents the features of the real-life situation with some level of assumptions and simplification; using mathematics and statistics to analyze the model and to draw conclusions; interpreting the results of the mathematics in the original situation; and validating the conclusion either to improve the model or lead to a report (CCSSI 2010). (See **fig. 3**.)

In the next section, we consider specific examples to describe the authenticity of modeling tasks and the application of the modeling cycle.

MODELING HIERARCHY

Three types of problems typically occur in curriculum materials: (1) word problems, (2) standard applications, and (3) true modeling problems (Niss, Blum, and Galbraith 2007). This hierarchy can be used as a framework for choosing appropriate tasks for mathematics classrooms. Let's look at examples of each type and consider how they are similar to and different from one another in regard to authenticity and the modeling cycle.

Word Problems

Word problems are merely the “‘dressing up’ of a purely mathematical problem in words referring to a segment of the real world” (Niss, Blum, and Galbraith 2007, p. 11). Consequently, the solution process includes a straightforward interpretation, as in this example:

Fig. 1 Modeling hierarchy classifies the first version as a word problem but the second version as a standard application.

The Six Criteria for Authentic Tasks

Characteristic	Description
Event	In a simulation of a real-life task, the event has taken place or has a fair potential of taking place.
Question	The question is one that actually might be posed in the event.
Purpose in the figurative context	The purpose of the task needs to be as clear to the students as it would be if they actually encountered it in context—either provided explicitly from the description or implicitly from the context.
Information or data	The task describes specific subjects, objects, and places, includes accessible data, and provides identical or close numbers and values as in the simulated situation.
Language use	The task does not, for example, include difficult terms that hinder the students in the solution process if the corresponding difficulties do not occur in the simulated situation.
Tool demands	The tools and guidance in using them have to be reasonably the same in the simulated problem and school situation; mathematical knowledge and skills that students need for solving the task are available.

Fig. 2 A description is provided for characteristics of authentic tasks (Palm 2008).

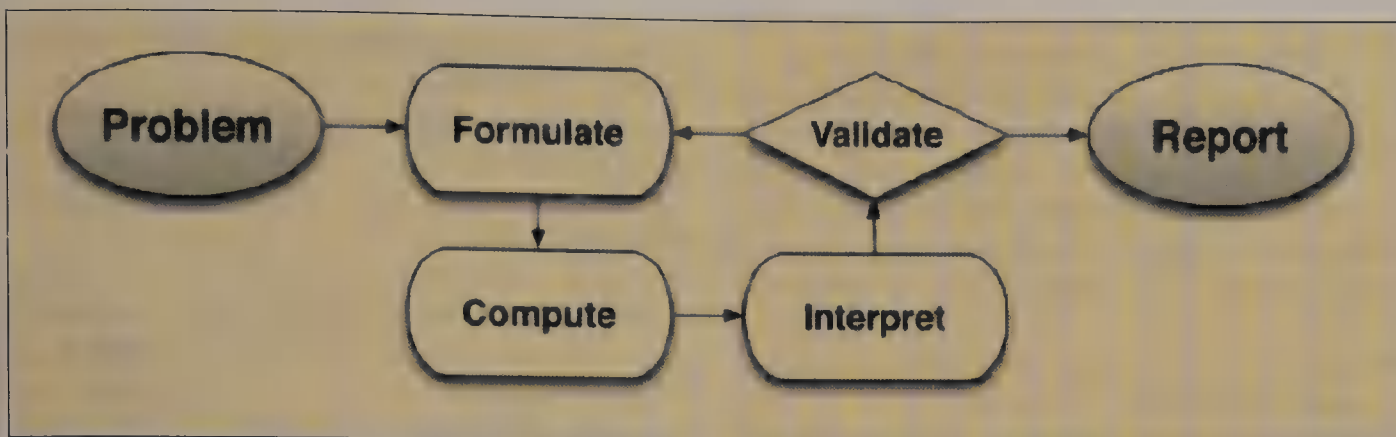


Fig. 3 The modeling cycle has several stages (CCSSI 2010, p. 72).

Michelle invests \$15,000 in a partnership that has four other partners. The total investment of all the partners is \$240,000. What percentage of the business does Michelle own?

This is a word problem because the context is not an integral part of the solution. Students need only take the numbers from the problem, write the fraction that represents Michelle's investment as a portion of the total investment, and convert the fraction to a percentage. Even though the problem is written in words, it uses a numerical algorithm that is not dependent on the context of the problem.

It might appear that this problem could be an authentic task because its context could actually occur and the question would be reasonable for this event. However, the problem lacks the complexity found in real-world situations of this type. Referring back to the modeling cycle, we see that there is little to formulate and that the results require little or no interpretation or validation.

Standard Applications

Standard applications are problems in which the solution strategy is immediately at hand "without further regard to the nature of the given real world context" (Niss, Blum, and Galbraith 2007, p. 12) and the translation of the problem's information to the mathematical analysis is relatively straightforward. Standard applications reflect some level of translating from the real-world context to the mathematical problem and thus are different from word problems. The second version of the Bus problem presented at the beginning of the article can be considered a standard application. The following example, Design a Staircase, is used to illustrate this type of problem.

The Design-a-Staircase Problem

The problem may be stated as in **figure 4**.

This is a standard application problem in which the context can be stripped away to expose a purely mathematical question about finding the slope

The Design-a-Staircase Problem

Your job is to design a staircase that has a total rise of 11 feet and meets these design guidelines:

1. The slope of the staircase must be between 0.55 and 0.85.
2. Twice the rise plus the run must be between 24 and 25 inches.
3. There can be no irregular steps.
4. Each step must be the same size.

How many risers and treads are there? What size are they? Explain your design and include any calculations. Show how each design guideline is met.

Some useful terms are *tread* (the horizontal part of a step); *run* (the length of the tread); *riser* (the vertical part of a step); *rise* (the height of the riser); and *slope* (a measure of the steepness of a staircase found by dividing the riser height [rise] by the tread length [run]). (Shannon 1999)

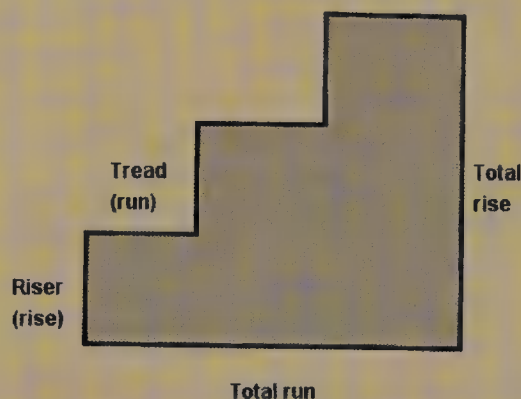


Fig. 4 This problem is classified as a standard application.

under prescribed constraints. When solving this task, students need more than the four arithmetic operations (for instance, conceptual knowledge of slope is required), setting it apart from the previous category, word problems. Terms such as *tread*, *run*, *riser*, and *slope* are illustrated for students who are not familiar with them. The task is accessible to students who have fundamental concepts about

The Design-a-Cola-R-Us-Can Problem

You have been hired by the Cola-R-Us Company to evaluate its current packaging. You are asked to determine how the can might be adjusted so that it requires as little aluminum as possible but still contains 330 ml of carbonated bliss.

Your report to Cola-R-Us should include

- The dimensions you recommend for efficient use of aluminum
- The calculations and work that support your recommendation
- A comparison between actual and ideal dimensions with an explanation as to why these might differ (Tran 2008)

Fig. 5 This example is a true modeling problem.

slope (*tool demand criterion*). In addition, students can use multiple problem-solving strategies, such as guess-and-test and making tables. Further, students can find multiple solutions using parameters or constraints to address the inequality. Even though this context has the potential to occur in real life and there is an explicit purpose for solving the problem, placing the context within a particular house design or other setting (*information criterion*) could enhance the problem.

This task addresses some aspects of the authenticity criteria. First, the context has potential for real-world settings, the questions raised are of interest in everyday life, and the students see the purpose when solving the tasks (*event, question, and purpose criteria*). However, the task is not situated in a specific simulation (*information or data criteria*). Second, the language used in the task is familiar (*language use criterion*). Any terms that might be unfamiliar to students are defined so that students have access to the problem. Finally, the tools used in the solution approach are reasonably accessible (*tools demand criterion*). Specifically, calculators would be appropriate tools in either the mathematical or the real-world setting. With regard to the modeling cycle, this problem does

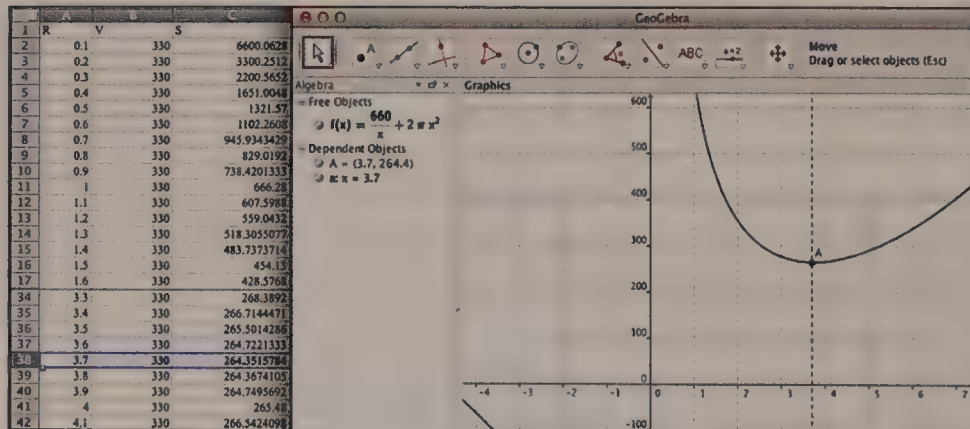


Fig. 6 A spreadsheet and a graph lead to the same solution.

not require an interpretation or validation of the results—that is, students do not have to specify critical variables (e.g., slope) to formulate a mathematical problem.

TRUE MODELING

True modeling problems involve the full cycle: Start with a question, then formulate a model, and finally solve, interpret, and validate within a mathematical and contextual situation. The following task is appropriate for high school students who know the formulas of surface area and volume of different solid shapes and know how to find maxima or minima of a function using graphing calculators or the derivative. We can proceed to determine the criteria in the authenticity framework and modeling cycle.

The Design-a-Cola-R-Us-Can Problem

The problem may be stated as in **figure 5**.

The Design-a-Cola-R-Us-Can problem is an example that requires students to integrate concepts and skills from two conceptual categories at the CCSSM high school level—functions (Interpreting Functions domain) and geometry (Modeling with Geometry domain)—to solve the problem and interpret its results within the problem context.

In this situation, as a first step, we would identify the variables needed to formulate a mathematical question. The critical variable to consider is the can's shape. Is it possible that the can's shape is different from a cylinder? If so, how would that affect the dimensions, volume, surface area, and the amount of aluminum required? Most students think of a cylinder and formulate the mathematical

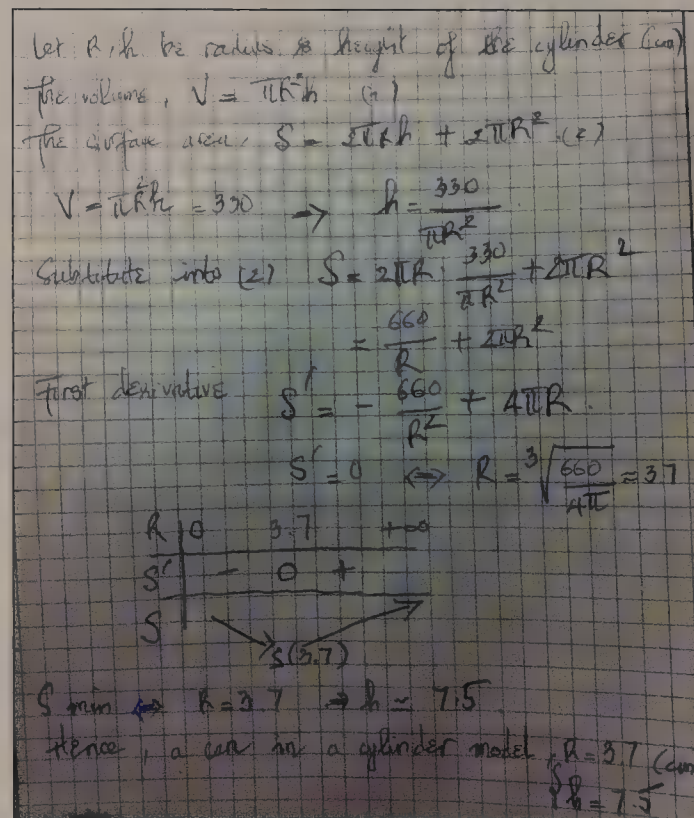


Fig. 7 Calculus leads to the same solution.

problem: Find a cylinder with the volume of 330 such that the total surface area is the minimum surface area possible.

Students should know the formulas for the volume and surface area of a cylinder (a CCSSM grade 8 learning expectation). If h and R are the height and radius of the cylinder (in cm), respectively, then the volume of the cylinder is $V = \pi R^2 h$, and the total surface area of the cylinder is $S = 2\pi R^2 + 2\pi R h$. The mathematical problem becomes this: Find R and h such that $V = 330$ and S is the minimum value.

Students can approach the solution in many ways. **Figures 6** and **7** show samples of preservice mathematics teachers' work using different methods, including creating an Excel® spreadsheet, graphing on the coordinate plane, and calculating the derivative to find maxima and minima. All solutions presented lead to values $h = 7.5$ and $R = 3.7$.

Interpreting the solution, the designer should specify a cylinder with a height of 7.5 cm and a radius of 3.7 cm to produce a can containing 330 ml and using the minimal amount of aluminum. A student may report this result as the best possible solution or may consider other models (e.g., spheres or different kinds of prisms) as part of the validating and reporting phases of the modeling cycle.

Regardless, a complete solution would include

an explanation as to why there might be a difference between actual and ideal dimensions, possibly including issues of both practical design and mathematics. For example, students using a cylinder model can see the differences in the dimensions of the existing can and their ideal can with minimum aluminum. Other factors to consider might be the palm width of customers and the aesthetics of the can. Among the solid shapes with the same volume, a sphere has the minimum surface area. However, if a sphere is used, other factors would need to be considered, such as the can's ability to stand up and the bundling of the cans.

A specific characteristic that makes this problem different from the standard applications is that the full process of modeling occurs. That is, students have to formulate a mathematical problem from a real-life situation. In particular, they have to choose a model for the can and interpret the phrase "contains 330 ml of carbonated bliss" to mean that the volume is 330 ml. Students must relate the requirement to use as little aluminum as possible to a requirement about the total surface area. Further, in their report, students have to consider real-life information to iterate the modeling process. The Design-a-Staircase and the Design-a-Cola-R-Us-Can problems share some similar

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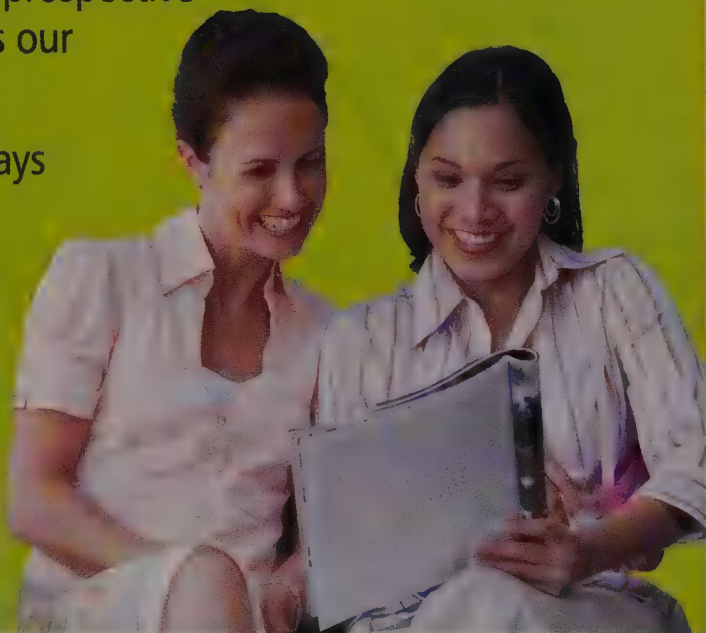
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characteristics. However, when solving the Design-a-Staircase problem, students do not have to cycle back to improve the model.

As we consider the Design-a-Cola-R-Us-Can problem, we see that this task met all six criteria in the authenticity framework. First, the event is realistic—designing the can has potential to take place in real life. Second, the question is of interest in everyday contexts because people tend to maximize benefit. Third, the purpose of the task is clear (that is, to find new dimensions so that production of the can efficiently uses the aluminum and then to compare the new and current cans). Fourth, the language is familiar to students. Fifth, the information is specific in the sense that it has a name, Cola-R-Us, and a specific volume, 330 ml (volume of the current can). Finally, the tools demand is appropriate for high school students; that is, students can use a derivative, spreadsheet, or graphing calculator to solve the problem.

RECOMMENDATIONS

Comparing the problem structures and the requirements for a solution leads us to recommendations supporting the selection of authentic problems that can follow the modeling cycle. Desired features of these problems include the following:

1. The expectation that mathematics beyond arithmetic operations and algorithms will be needed to solve the problem, including an expectation of distinguishing relevant and irrelevant information
2. The potential to occur in everyday or realistic contexts
3. Multiple solutions and solution processes, presented with justification (Greer, Verschaffel, and Mukhopadhyay 2007)

Selecting tasks that incorporate modeling and the authenticity criteria allows us to emphasize the Standards for Mathematical Practice in CCSSM (CCSSI 2010). In mathematical modeling tasks, we can observe students' performance as they make sense and persevere in solving the problem, choose variables to represent the problem, reason abstractly and quantitatively, move back and forth between contextualized and decontextualized settings, and develop models with appropriate mathematical or real life tools (CCSSI 2010).

The frameworks of authenticity and modeling hierarchy as well as the modeling cycle may help teachers determine which problems from the curriculum materials would best support students' learning. Seeing the utility of the mathematics may help students develop a positive attitude about mathematics and increase their engagement in significant and complex problems.

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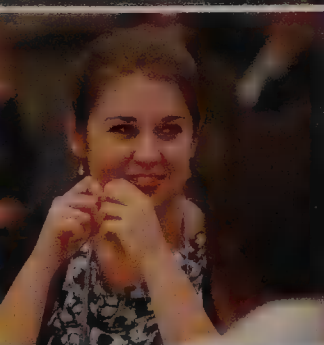
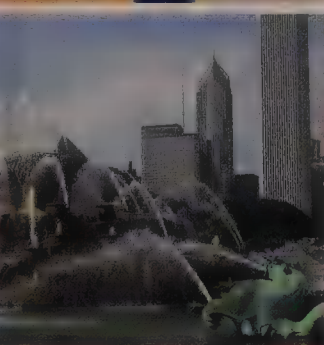
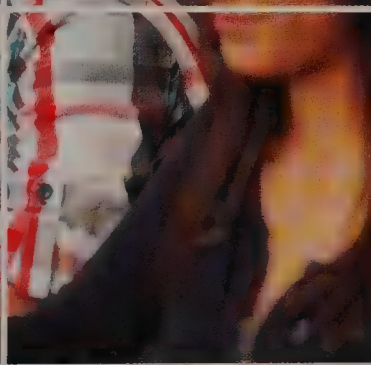
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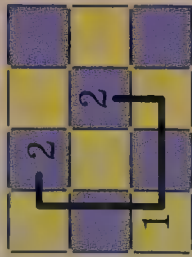
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The editors wish MT readers a restful vacation. If your summer travels inspire a mathematics problem, please share it with us!

A knight living on a 3×4 checkerboard is planning a tour to visit each of the 12 squares exactly once. A legal move must be L-shaped: two vertical steps followed by one horizontal step or two horizontal steps followed by one vertical step. Find at least one route for the knight.



Find the height of an equilateral triangle if the area (in square units) and the perimeter (in linear units) are the same.

4

Mount McKinley (20,320 ft.) is the highest point in Alaska and the United States. Britton Hill (345 ft.), in Florida, is the lowest of the high points of the 50 states. The mean of these 50 highest points is 6162 ft. Eleanor objected, pointing out that the District of Columbia was not represented. Find the new mean when Reno Reservoir (415 ft.) in Washington, D.C., is included.

A circle has center A and equation $x^2 + y^2 - 6x = 12y + 124$. Two lines, l_1 and l_2 , intersect at B ; their equations are $4x + 3y = 30$ and $3x - 4y = 10$, respectively. The circle and l_2 intersect in quadrant III at point C . Find the area and perimeter of $\triangle ABC$.

5

Find the apothem of a regular hexagon if its perimeter and its area are numerically equal.

8

Prove that if a polygon is circumscribed about a circle of radius 2, then the numerical values of the area and perimeter of the polygon are equal.

A group is traveling to New York City to visit 3 museums. Everyone wants to see at least 1 museum, and $2/3$ plan to see at least 2. Five choose only MoMA, and $1/5$ choose only MoMath. One-third choose MoMath and MoMA, $1/4$ choose the Met and MoMA, and $1/4$ choose the Met and MoMath. If fewer than 100 people make the trip, how many will visit the Met?

2

If a right triangle has sides $4x$, $4x + 2$, and $4x - 2$, how would you find x without using a calculator or extended calculation?

6

The equilateral triangle obtained in the problem for May 8 and the regular hexagon obtained in the problem for May 9 share the characteristic that their perimeters and areas are numerically equal. Find the radius of the inscribed circle for each polygon.

10

The Great Pyramid of Giza is a square pyramid, originally about 482 ft. in height, with base edges of 756 ft. If we call the height of the pyramid h , the slant height l , and the apothem of the base a , compute to two decimal places the ratios $h:a$, $l:h$, and $l:a$.

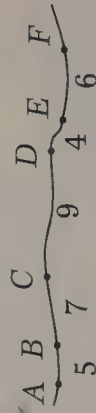
The three positive integers 1, 25, and 49 are in an arithmetic progression (in this case, with common difference 24), and their product is a perfect square. Is there another example of three positive integers in an arithmetic progression whose product is a perfect square?

3

Prove that if the sides of a right triangle are in arithmetic progression, then the triangle is similar to a 3-4-5 triangle.

7

Six friends live near the same highway, separated by the distances shown. The highway is the best way to travel there, and its route can be modeled satisfactorily by a line. Find the best meeting place for the six if the chosen location must minimize the total distance traveled.



11

Find a right triangle such that the sides are in geometric progression, and find the following extended ratio:

short leg : long leg : hypotenuse

Give both exact values and values rounded to three decimal places. When you finish, look at the result of the May 14 problem.

Recall the data from the problem of May 1. The standard deviation of the highest points in each of the 50 states is 5035 ft. When Reno Reservoir (415 ft.) in Washington, D.C., is included, will the standard deviation increase or decrease? What is the new standard deviation?

Theon's ladder (see the problem of May 19) may have been used by Greek mathematicians to find better and better rational approximations to the square root of 2. Confirm that the ratio of integers on each rung approximates $\sqrt{2}$. On what rung does the ratio approximate $\sqrt{2}$ with an error of no more than 0.0001?

If

$$b:c=7 \text{ and } a:b=3,$$

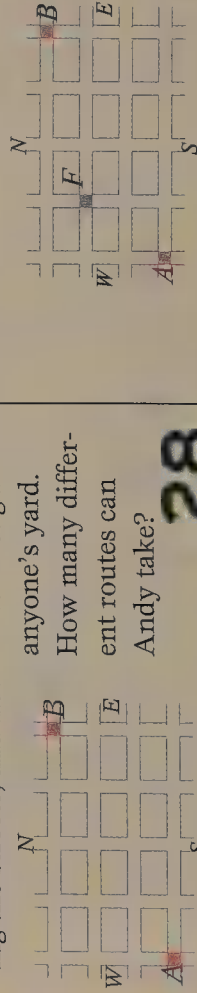
$(a-b):(b-c).$

find

Andy and Bob live in a town where the streets intersect at right angles. When Andy walks to Bob's house (A to B on the grid), he walks north or east only, following the streets, and never cuts through anyone's yard.

How many different routes can Andy take?

Suppose that the intersection marked F has been blocked off and Andy must avoid it (see the problem for May 28). How many paths does he have to choose from now?



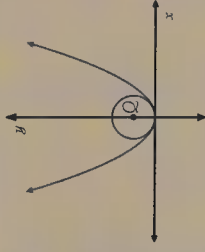
Find all values of m such that the line l with equation

$$y = mx + 3$$

is tangent to the parabola

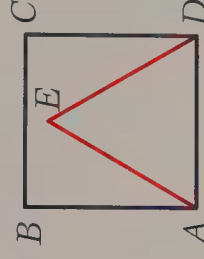
$$f(x) = \frac{(x-1)^2}{2} + 4.5.$$

A parabola with the equation $y = x^2/8$ is tangent to the x -axis at the origin. Circle Q with center on the positive y -axis is also tangent to the x -axis at $(0, 0)$. What is the largest possible radius for circle Q such that the parabola and the circle have only the point $(0, 0)$ in common?



A square, $ABCD$, has M as the midpoint of $\overline{AB}$. Construct, using a straightedge and compass (or a drawing program), the circle that is tangent to $\overline{AB}$ at M and passes through C and D . If a side of the square has length 4, what is the radius of this circle?

Write the odd integers in order. The first number after 1 is 3, so strike out every 3rd integer in the list. Your list becomes 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, ... The next smallest number is now 7; strike out every 7th number from the list. Continue in this manner. The numbers that remain are called *lucky numbers*. Find four ways to write 34 as the sum of two lucky numbers.



Equilateral triangle AED has been constructed on the side of square $ABCD$. If the area of the square is 16, what is the area of $\triangle AED$?

For what values of x is the expression

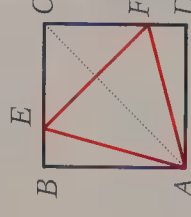
$$\log_{\alpha}(\log_{\alpha}(\log_{\alpha} x))$$

defined?

Equilateral triangle *AED* has been

constructed on the side of square $ABCD$. If the area of the square is 16, what is the area of $\triangle AED$?

Find a and b such that


$$a^0 + a^1 + a^2 = 31 = b^0 + b^1 + b^2 + b^3 + b^4.$$

Thus, 31 can be written in two ways as the sum of at least three successive powers beginning with exponent zero. The only other number known to share this property is 8191. Find the two ways to write 8191.

Find the value of x if

$$g^x - g^{x-1} = 1944.$$

Taejas is playing a board game that requires him to use both the sum and the product of the upper faces on three standard dice. He needs a sum of at least 6 on his next toss to move his token.

If the sum of the upper faces is exactly 6, what is the probability that the product is also 6?

Equilateral triangle AEF has been constructed in square $ABCD$ such that vertices E and F lie on $\overline{BC}$ and $\overline{CD}$ respectively. If the area of the square is 16, what is the area of $\triangle AEF$?

Find a and b such that

Thus, 31 can be written in two ways as the sum of at least three successive powers beginning with exponent zero. The only other number known to share this property is 8191. Find the two ways to write 8191.

SOLUTIONS to calendar

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1. About 6049 ft. To find the new mean, find the sum of the heights of the 50 highest points first: $50 \cdot 6162 = 308,100$. Add the additional observation and divide by 51: $(308,100 + 415)/51 \approx 6049$ ft.

2. 28. The fractions given in the problem— $2/3$, $1/4$, and $1/5$ —imply that the

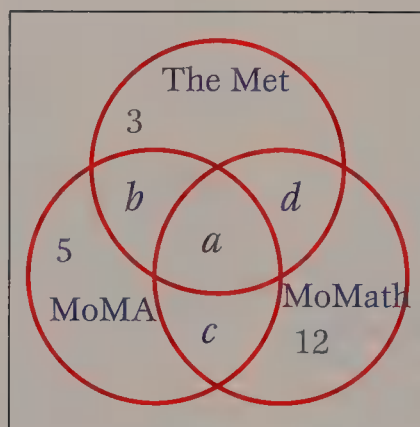
Edited by **Margaret Coffey**, Margaret.Coffey@fcps.edu, Thomas Jefferson High School for Science and Technology, Alexandria, VA 22312, and **Art Kalish**, artkalish@verizon.net, Syosset High School (retired), Syosset, NY 11791

The Editorial Panel of *Mathematics Teacher* is considering sets of problems submitted by individuals, classes of prospective teachers, and mathematics clubs for publication in the monthly Calendar. Send problems to the Calendar editors. Remember to include a complete solution for each problem submitted.

Problems 5, 6, and 8 were submitted by Richard Forringer of Durham Academy (retired), North Carolina.

Other sources of problems in calendar form available from NCTM include *Calendar Problems from the "Mathematics Teacher"* (a book featuring more than 400 problems, organized by topic; stock number 12509, \$22.95) and the 100 Problem Poster (stock number 13207, \$9.00). Individual members receive a 20 percent discount off this price. A catalog of educational materials is available at www.nctm.org.—Eds.

number of travelers is divisible by 3, 4, and 5. Sixty is the only such number less than 100. Since $2/3$ of the group will visit at least two museums and everyone will visit at least one, we know that $1/3$ of the group, or 20 people, will visit exactly one museum. Five of the 20 will visit MoMA only, and $1/5$ of the group, or 12 people, will visit MoMath only, so we conclude that 3 will visit the Met only. Show the number of people represented in each region of the Venn diagram. Then $c + a = 20$, $b + a = 15$, $d + a = 15$, and $a + b + c + d = 40$. The second and third equations imply that $b = d = 15 - a$. Rearrange the first equation and substitute into the fourth: $a + 2(15 - a) + (20 - a) = 40 \rightarrow a = 5$. Therefore, the number of travelers who plan to visit the Met is $3 + 10 + 5 + 10 = 28$.

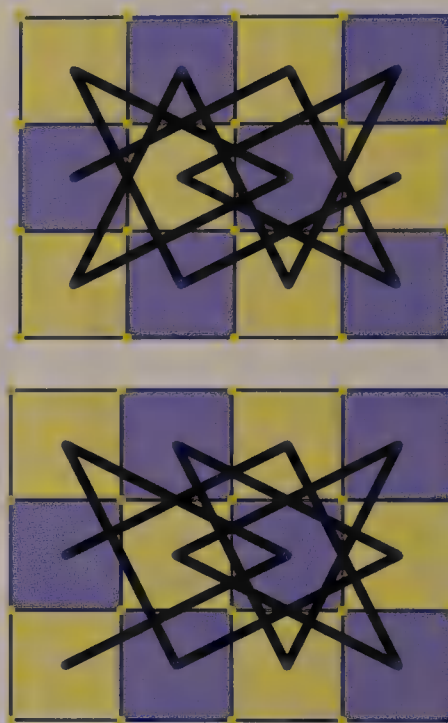


Because of space constraints, the abbreviated names of the three museums were used in the problem. The Met is the Metropolitan Museum of Art; MoMA is the Museum of Modern Art; and MoMath is the National Museum of Mathematics.

3. Send student solutions to the Problem of the Month editors, Seán Madden (smadden1@greeleyschools.org) or Ricardo Diaz (Ricardo.Diaz@unco.edu).

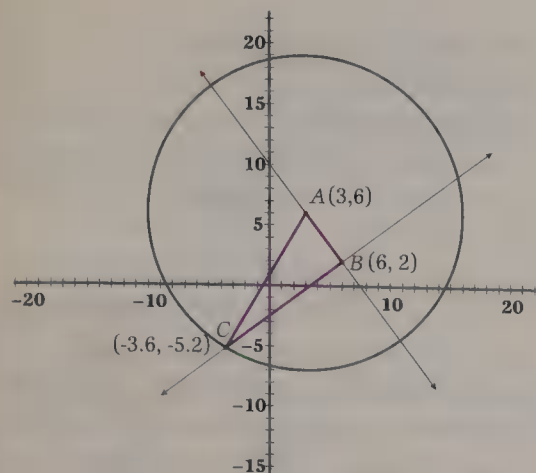
4. Two open tours are shown (see col. 3). A closed tour, in which the

knight begins and ends on the same square, is not possible on a 3×4 board.



Interested readers can find a more detailed discussion of knight's tours along with other fascinating chessboard problems in *Across the Board: The Mathematics of Chessboard Problems* (Watkins 2012); the first chapter is accessible online at <http://press.princeton.edu/chapters/s7714.pdf>.

5. Area 30, perimeter 30. Find the coordinates of B by solving the linear system to obtain $(6, 2)$. Complete the squares in the equation for circle A to obtain $(x - 3)^2 + (y - 6)^2 = 13^2$. Point A has coordinates $(3, 6)$ and lies on l_1 because $4(3) + 3(6) = 30$. Observe that l_1 with slope $-4/3$ and l_2 with slope $3/4$ are perpendicular. Thus, $\triangle ABC$ is a right triangle with hypotenuse $AC = 13$, and $AB = \sqrt{(3 - 6)^2 + (6 - 2)^2} = 5$. Apply the Pythagorean theorem to find that $BC = 12$. The area of $\triangle ABC$ is $(1/2) \cdot (5 \cdot 12) = 30$ square units, and the perimeter is 30 linear units (see col. 1, p. 683).



Alternate solution: If we did not observe that A lies on l_1 and that $l_1 \perp l_2$, we could find the coordinates of C by solving this system: $(x - 3)^2 + (y - 6)^2 = 13^2$ and $3x - 4y = 10$. Determinants or other methods can be used to find the area of $\triangle ABC$.

6. 2. Order the three expressions from smallest to largest, and note that the expressions are in arithmetic progression, with common difference 2. The only right triangle whose sides all differ by 2 is a 6-8-10 right triangle. Therefore, $4x$, the middle value, must be 8, and x must be 2.

7. Let the three sides be represented by $x - a$, x , and $x + a$, where a is the common difference. Since we are given a right triangle, we know that $(x - a)^2 + x^2 = (x + a)^2 \rightarrow x^2 - 2ax + a^2 + x^2 = x^2 + 2ax + a^2 \rightarrow x^2 = 4ax$. Reject $x = 0$ and accept $x = 4a$. Substitute into the original expressions to obtain the side lengths $3a$, $4a$, and $5a$.

8. 6 units. Use the formula for the area of a regular polygon: $area = 0.5ap$, where a is the apothem and p is the perimeter. Since $area$ and $perimeter$ must have the same numerical value, we have $p = 0.5ap$, so $a = 2$. The apothem of an equilateral triangle is $1/3$ the height, so the height must be 6.

Alternate solution: Let s be the side length of the triangle. Then the perimeter is $3s$, and the area is $s^2\sqrt{3}/4$. Since these two expressions must be equal, $s = 0$ or $s = 12/\sqrt{3}$. Use the 30-60-90° relationship to find the height: $h = s\sqrt{3}/2 = 6$ units. Confirm that the perimeter and the area are numerically equal:

$$3s = 3(12/\sqrt{3}) = 12\sqrt{3}$$

and

$$s^2\sqrt{3}/4 = (12/\sqrt{3})^2(\sqrt{3}/4) = 12\sqrt{3}.$$

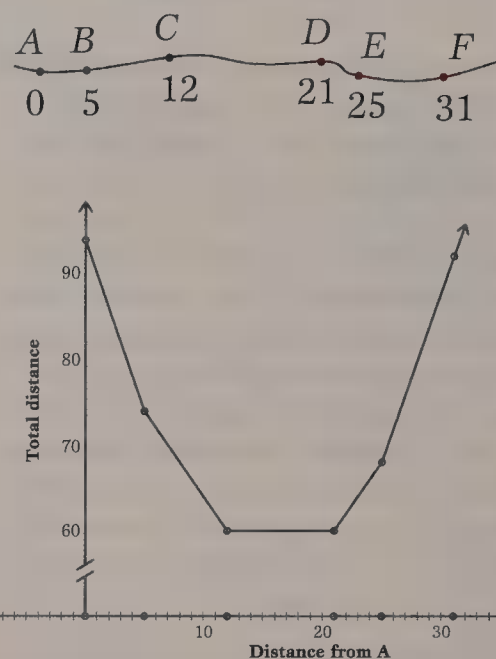
9. 2 units. Let s be a side of the regular hexagon and let a be the apothem. Then the perimeter must be $6s$, and the area must be $ap/2 = a \cdot 3s$, implying that $a = 2$.

10. 2. For any regular polygon, the apothem equals the radius of the inscribed circle. In the case of the hexagon, we already know the apothem, so the radius of the inscribed circle is 2. The apothem of an equilateral triangle is one-third the height, a consequence of the 30-60-90° relationship. We can also obtain this result from the following theorem: The centroid of a triangle divides each median in the ratio 2:1. The height of the equilateral triangle was 6, so the apothem is 2. We find that the radius of the inscribed circle is 2 in both polygons.

11. Point C , point D , or any point between C and D (see col. 3). Suppose that everyone met at person A 's house. Then the total distance traveled would be $5 + 12 + 21 + 25 + 31 = 94$ miles. Consider meeting 1 mile east of point A . Then A must travel 1 additional mile, but each of the other five friends travels 1 mile less, for a net reduction of 4 miles. As the proposed meeting place moves east, this rate of change remains constant until the meeting place reaches B 's house. Here the total distance traveled is $94 - 4(5) = 74$ miles. If the meeting place is 1 mile east of B 's house, both A and B travel an additional mile, but the other four friends each travel 1 mile less, for a net reduction of 2 miles. Since C lives 7 miles east of B , the total distance traveled if the meeting place is C 's house will be $74 - 2(7) = 60$ miles. Consider moving the meeting place farther east. Each extra mile traveled by A , B , and C will be exactly offset by fewer miles traveled by D , E , and F until they reach D 's house. Confirm that the total distance traveled if the meeting place is D 's house is also 60 miles. If the meeting place is moved east of D 's house, there will be a net

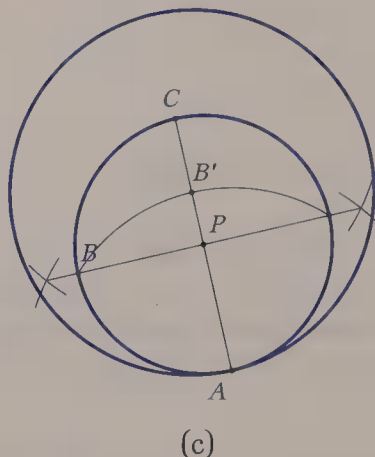
increase of total distance traveled. Thus, meeting at C 's house or D 's house or any point in between results in a minimum of 60 miles.

It is helpful to locate the points as distances from an origin—say, point A —so that we can plot total distance traveled against the meeting place location. The rates of change mentioned in the discussion above appear as the slopes of the piecewise-linear function.



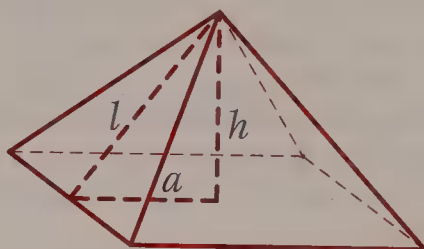
12. Consider first the case of a regular polygon circumscribed about a circle of radius 2. The circle will be tangent to the sides of the polygon at the midpoint of each side, so the polygon will have an apothem of length 2. With apothem a and perimeter p , the polygon has area $ap/2 = 2p/2 = p$.

Consider next the general case of any convex polygon circumscribed about a circle of radius r (see fig. at top of p. 684). When segments are drawn from the center to each of the vertices, the polygon is divided into a series of triangles. The height of each triangle is the radius of the inscribed circle, r . Since tangents to a circle from a point are congruent, the perimeter is composed of two sets of congruent segments, $2(a + b + c + d + \dots + n)$, as shown in the figure. The area of the polygon is the sum of the areas of these triangles: $(1/2) \cdot (a + b)r + (1/2)(b + c)r + (1/2)(c + d)r + \dots + (1/2)(n + a)r = (r/2)(2a + 2b + 2c + 2d + \dots + 2n) = r(a + b + c + d + \dots + n)$. Since $r = 2$ is given, the numerical values of the perimeter and the area are equal.



13. If the ratio of areas is 2 : 1, then the ratio of radii is $\sqrt{2}$: 1. This ratio of lengths appears as the hypotenuse of an isosceles right triangle to its leg. First construct a circle P and its diameter. The perpendicular bisector of the diameter gives a diameter AC (see fig. a). Let radius $PA = 1$; then $PB = 1$ and $AB = \sqrt{2}$. Mark a point B' on diameter AC so that AB' equals AB (see fig. b). Finally, construct a circle with center B' and radius $B'A$ (see fig. c).

14. $h:a = 1.28$; $l:h = 1.27$; and $l:a = 1.62$. The height h , slant height l , and apothem of the base a form a right triangle. Use the Pythagorean theorem to find the slant height: $l^2 = 482^2 + (756/2)^2 \rightarrow l \approx 613$ ft. So $h:a = 482/378 = 1.28$; $l:h = 613/482 = 1.27$; and $l:a = 613/378 = 1.62$.



l is slant height
 h is height
 a is apothem of base

15. $1:\sqrt{\phi}:\phi$, or $1:1.272:1.618$. Yes. We have a right triangle with legs a and b and hypotenuse c . If the sides are in geometric progression, then $a:b = b:c \rightarrow b^2 = ac$. We have $c^2 - ac - a^2 = 0$. Considering this expression a quadratic in c , we obtain

$$c = \frac{a \pm a\sqrt{5}}{2} \rightarrow \frac{c}{a} = \frac{1 \pm \sqrt{5}}{2}$$

(we drop the negative because $c/a > 0$). This expression is the golden ratio $\phi \approx 1.618$. Since $c:a$ is ϕ (or $\phi:1$) and since b is the geometric mean between 1 and ϕ , then $b = \sqrt{\phi} \approx 1.272$. These values are the same as those obtained from the right triangle in the cross section of the Great Pyramid of Giza. Some will claim that this value is not a coincidence. The editors, however, rely on the scholarship of B. L. van der Waerden, Dirk Jan Struik, and Sir Thomas Little Heath; see Eli Maor's *Pythagorean Theorem: A 4,000-Year History* (2010, p. 15). The consensus of these scholars is that we

have absolutely no evidence that the Egyptians knew the Pythagorean theorem, so they could not have deliberately constructed a right triangle whose sides are in geometric progression.

16. Standard deviation will increase; 5049 ft. At 415 ft. above sea level, the District of Columbia's highest point must be fairly close to the low extreme of the distribution. (In fact, only Florida's highest point is lower.) The absolute deviation from the mean for this observation, $|415 - 6049| = 5634$, is quite a bit larger than the original value for σ . So we can guess that the standard deviation will increase rather than decrease.

The main component of standard deviation calculation is the sum of the squared deviations:

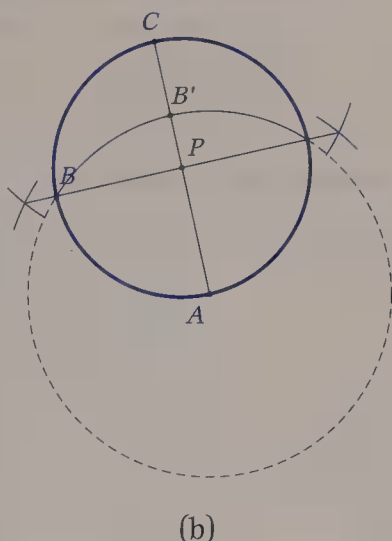
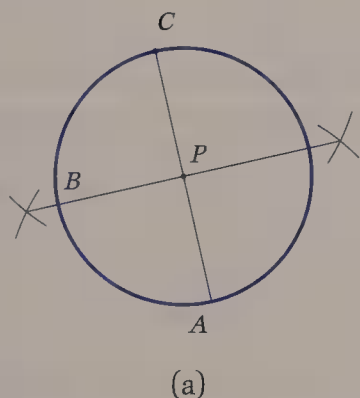
$$\sigma = \sqrt{\frac{\Sigma(X - \mu)^2}{N}} \rightarrow \Sigma(X - \mu)^2 = N\sigma^2.$$

For $N = 50$, we had $\mu = 6162$ and $\sigma = 5035$ with $\Sigma(X - \mu)^2 = 50(5035)^2$.

Because we want to find the standard deviation when $N = 51$ and $\mu = 6049$, we will need to total 51 terms (instead of 50) and account for the change in the mean. Noticing that $6049 = 6162 - 113$, we write the following:

$$\begin{aligned}
 & \sum_{i=1}^{51} (x_i - 6049)^2 \\
 &= \sum_{i=1}^{50} (x_i - 6049)^2 + (x_{51} - 6049)^2 \\
 &= \sum_{i=1}^{50} (x_i - 6162 + 113)^2 + (415 - 6049)^2 \\
 &= \sum_{i=1}^{50} ((x_i - 6162)^2 + 2 \cdot 113(x_i - 6162) + 113^2) \\
 &\hspace{15em} + 31,741,956 \\
 &= \sum_{i=1}^{50} (x_i - 6162)^2 + 2 \cdot 113 \sum_{i=1}^{50} (x_i - 6162) \\
 &\hspace{15em} + \sum_{i=1}^{50} 113^2 + 31,741,956
 \end{aligned}$$

In the last line, the first term equals the original $\Sigma(X - \mu)^2$. The second term is zero because the sum of the deviations from the mean is always zero. The third term is a sum of 50 identical constants. Thus, the calculation gives



$$\sum_{i=1}^{51} (x_i - 6049)^2 = 50(5035)^2 + 0$$

$$+ 50 \cdot 113^2 + 31,741,956 = 1,299,941,656.$$

Divide by 51 and take the square root to obtain $\sigma = 5049$ ft., to the nearest foot. Source for statistics: <http://www.peakbagger.com/list.aspx?lid=12004>

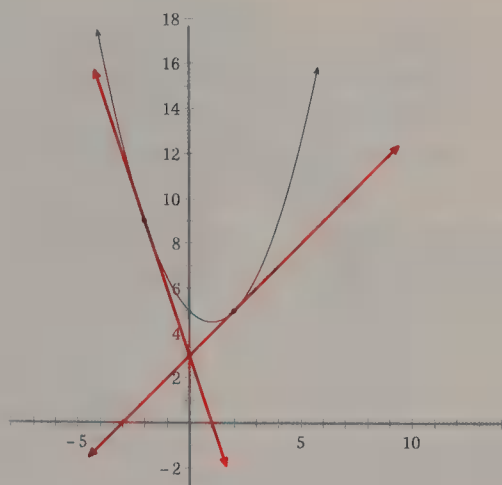
17. -3 or 1. We expect two values of m , one positive and one negative, because we see that there will be a point of tangency in Quadrant I and another in Quadrant II. We have a linear function and a quadratic; set them equal and solve for x :

$$mx + 3 = (x - 1)^2/2 + 4.5$$

$$\rightarrow x^2 - (2m + 2)x + 4 = 0$$

$$\rightarrow x = \frac{2m + 2 \pm \sqrt{(2m + 2)^2 - 4(4)}}{2}$$

Since each of the two lines through $(0, 3)$ must intersect the parabola in exactly one point, we set the discriminant equal to 0: $(2m + 2)^2 - 16 = 0 \rightarrow 2m + 2 = \pm 4 \rightarrow m = -3$ or 1.



18. 3.5. Begin by factoring the left side: $9^{x-1}(9 - 1) = 1944 \rightarrow 9^{x-1} = 243$. Rewrite both sides using 3 as the base:

$$(3^2)^{x-1} = 3^{2x-2} = 3^5.$$

So $2x - 2 = 5$, giving $x = 3.5$.

19. 169, 239. Call the number pair on the n th row x_n, y_n . We see that for $n > 1$ we have $x_n = x_{n-1} + y_{n-1}$ and $y_n = x_n + x_{n-1}$. Using this rule, we have the pair 29 and 41 on rung 5, the pair 70 and 99 on rung 6, and the pair 169 and 239 on rung 7.

20. 6th rung. The square root of 2 is approximately 1.414213562. The first 7 rungs of Theon's ladder give us $r_1 = 1/1 = 1$; $r_2 = 3/2 = 1.5$; $r_3 = 7/5 = 1.4$; $r_4 = 17/12 = 1.41\bar{6}$; $r_5 = 41/29 \approx 1.4138$; $r_6 = 99/70 = 1.41\bar{4}2857$; and $r_7 = 239/169 \approx 1.414201$. The difference between $\sqrt{2}$ and 41/29 is about 0.00042, whereas the difference between 99/70 and $\sqrt{2}$ is about 0.000072. So we need the pair of integers on the 6th rung. Note that this approximation uses only two-digit numbers.

21. 4. Since circle Q is tangent to the x -axis at $(0, 0)$, it will have center $(0, r)$, where r is the radius of the circle. Its equation must be $(x - 0)^2 + (y - r)^2 = r^2$. Substituting $8y$ for x^2 yields $8y + y^2 - 2ry = 0 \rightarrow y^2 + (8 - 2r)y = 0$. Since we must have exactly one solution, we set the discriminant of this quadratic equal to 0: $(8 - 2r)^2 - 4(0) = 0 \rightarrow r = 4$.

22. $(8, \infty)$. The expression $\log_8(a)$ is defined for $a > 0$, so $a = \log_8(\log_8 x) > 0$. If $\log_8(b) > 0$, then $b > 1$, implying that $b = \log_8 x > 1$. This final inequality implies that $x > 8$.

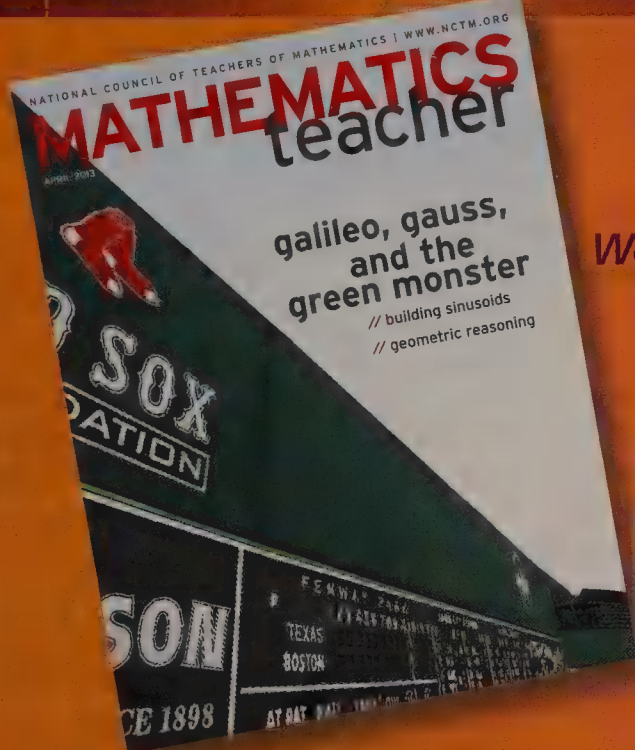
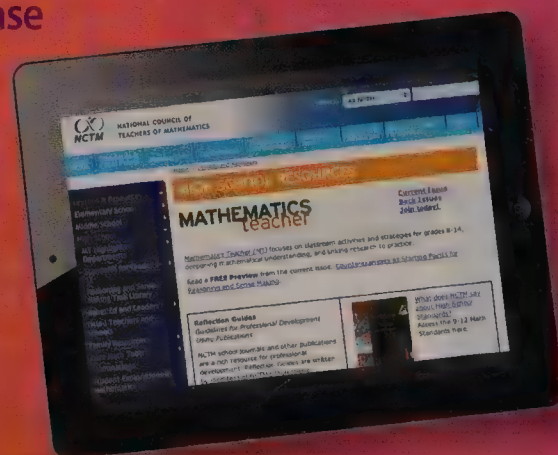
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23. $3/5$. There are three combinations of the dice that give a sum of 6: $\{4, 1, 1\}$, $\{1, 2, 3\}$, and $\{2, 2, 2\}$. There is only one way to roll three 2s, but there are three ways to roll 4 and two 1s, and there are six different orderings of 1, 2, 3, for a total of ten ways to roll the sum. The product of the dice will be 6 if the outcome is one of the orderings of $\{1, 2, 3\}$, so the probability that the product is 6, given that the sum is 6, is $6/10 = 3/5$.

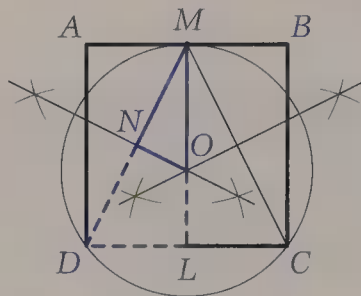
24. $7:3$. To obtain an equation in b alone, use $a = 3b$ and $b = 7c \rightarrow c = b/7$. Substituting gives $(3b - b):(b - b/7) = (2b):(6b/7) = 7:3$.

Alternate solution: The ratio $a:b = 3:1$ implies that $a:b = 21:7$. Therefore, $a:b:c = 21:7:1$. Then $(a - b):(b - c) = 14:6 = 7:3$.

25. 2.5. Let O be the center of the circle, and let L be the midpoint of $\overline{DC}$. The center of the circle must be equidistant from M , C , and D , so we construct the perpendicular bisectors of $\overline{MC}$ and $\overline{MD}$; use their intersection as the center of the circle. The circle circumscribes $\triangle MCD$.

If the square has side length 4, then $\triangle MLD$ in the figure has side lengths 4, 2, and $2\sqrt{5}$. With N as the midpoint of $\overline{MD}$, we have $MN = MD/2 = \sqrt{5}$. Because $\triangle MNO$ is similar to $\triangle MLD$, we can find the radius MO from a proportion $MO:MN = MD:ML \rightarrow$

$$\frac{MO}{\sqrt{5}} = \frac{2\sqrt{5}}{4} \rightarrow MO = 2.5.$$



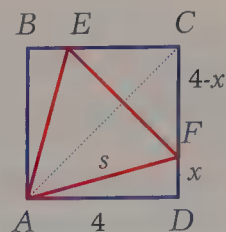
Alternate solution: Let O be the center of the circle, and let L be the midpoint of $\overline{DC}$, as shown in the figure. If $OM = OC = r$, then $OL = ML - OM = 4 - r$, and $LC = 2$. Use the Pythagorean theorem to solve for r : $OL^2 + LC^2 = OC^2 \rightarrow (4 - r)^2 + 2^2 = r^2 \rightarrow r = 2.5$.

26. $4\sqrt{3} \approx 6.928$. If the square has area 16, then a side length is 4. Use the Pythagorean theorem or the formula for the area of an equilateral triangle, given a side:

$$\frac{s^2\sqrt{3}}{4} = 4\sqrt{3}.$$

27. $16(2\sqrt{3} - 3) \approx 7.426$. Diagonal AC is a line of symmetry for the triangle and the square. A symmetry argument allows us to establish that $\triangle ECF$ is an isosceles right triangle. Let $FD = x$; then $CF = 4 - x$. Let s be the side length of $\triangle AEF$; then s is also the hypotenuse of $\triangle ECF$. In the $45\text{-}45\text{-}90^\circ$ triangle, we have $s = FC \cdot \sqrt{2} = (4 - x)\sqrt{2}$, so $s^2 = 2x^2 - 16x + 32$. In $\triangle FDA$, $AD^2 + FD^2 = AF^2 \rightarrow s^2 = 4^2 + x^2$. Equating for s^2 , we get $4^2 + x^2 = 2x^2 - 16x + 32 \rightarrow x = 8 - 4\sqrt{3}$ since $x < 4$. Therefore, $CF = 4 - x = 4\sqrt{3} - 4$, and $s = 4\sqrt{6} - 4\sqrt{2}$. Substitute this value into the formula for area and simplify:

$$\begin{aligned} \frac{s^2\sqrt{3}}{4} &= (4\sqrt{6} - 4\sqrt{2})^2 \cdot \frac{\sqrt{3}}{4} \\ &= 16(2\sqrt{3} - 3) \\ &\approx 7.426. \end{aligned}$$



Alternate solution 1: Triangles ADF and ABE are congruent by the hypotenuse-leg theorem, and, therefore, $FD = BE$ and $FC = EC$. Proceed as above.

Alternate solution 2: Since diagonal AC is a line of symmetry, $m\angle FAD$ is 15° . We can then use trigonometry to find that $s = 4/\cos(15^\circ)$ and proceed to find the area of $\triangle AEF$.

28. 35 routes. Andy will walk 7 blocks to reach Bob's house. We can model his path as an arrangement of 3 Ns for north and 4 Es for east. The number of paths is the number of ways to choose 3 of the 7 letters to be the Ns: ${}_7C_3 = 35$.

29. 23 routes. There are ${}_3C_2 = 3$ ways to walk from A to F and ${}_4C_1 = 4$ ways to walk from F to B . So there are $3 \cdot 4 = 12$ paths from A to B that pass through F , and there are $35 - 12 = 23$ paths from A to B that avoid intersection F .

30. $1 + 33$, $3 + 31$, $9 + 25$, and $13 + 21$. Follow the procedure to find the lucky numbers less than 34: 1, 3, 7, 9, 13, 15, 21, 25, 31, and 33. Four pairs from this list sum to 34.

31. $a = 5$, $b = 2$. The number $8191 = 2^0 + 2^1 + 2^2 + \dots + 2^{12} = 90^0 + 90^1 + 90^2$. The inequality $b < a < 6$ gives us a bound on possible bases, and we do not have many possibilities to consider. With powers of 2, we get $1 + 2 + 2^2 + 2^3 + 2^4 = 31$. Using three terms, we have $1 + 5 + 5^2 = 31$. We find the first solution for 8191 by checking the powers of 2: $2^{13} = 8192$, so we have $2^0 + 2^1 + 2^2 + \dots + 2^{12} = 8192 - 1 = 8191$. Looking for a sum with three terms, we find $1 + 90 + 90^2 = 8191$. For those interested in number theory, $31 = 2^5 - 1$ and $8191 = 2^{13} - 1$ are both Mersenne primes.

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2016 Focus Issue

Teaching Mathematics Online

Today's students live in a world of computers, tablets, smartphones, and cloud computing. The pace of innovation continues to accelerate. How can mathematics teachers capitalize on these trends for the good of our students? The Editorial Panel of *Mathematics Teacher* invites teachers, teacher educators, education researchers, and others to share their experiences in online mathematics education. Submissions that highlight how online technologies can support NCTM's Equity Principle are especially encouraged. In some cases, print alone will not adequately capture the total impact of the article; in such cases, having active links or other dynamic resources will be optimal.

All possible innovative uses of online resources cannot be enumerated here, but the following questions will get the conversation started.

OPPORTUNITY

How can online technologies—

- be leveraged for mathematics learning?
- provide for student collaboration across distances?
- improve professional development opportunities?
- affect the achievement gaps between different groups of students?

IMPLEMENTATION

How do you—

- remain connected to your students when teaching online?
- use specific online tools to help students learn?
- balance and complement traditional classroom formats with online technologies?
- compare the effectiveness of different methods for student learning?

These are just a few of our ideas; we want to know what you think. We particularly encourage teachers to share their own favorite online lessons.

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Equivalent Expressions Using CAS and Paper-and-Pencil Techniques

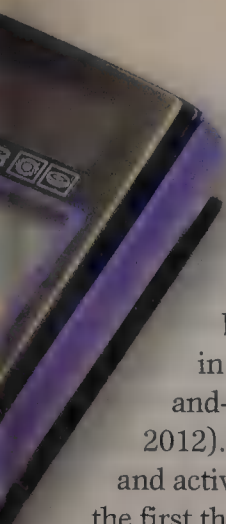
Designing activities to reconcile multiple representations supports students' focus and fluency.

Nicole L. Fonger

How can the key concept of equivalent expressions be addressed so that students strengthen their representational fluency with symbols, graphs, and numbers? How can research inform the synergistic use of both paper-and-pencil analysis and computer algebra systems (CAS) in a classroom learning environment? These and other related questions have important implications for any beginning algebra course. This article focuses on two approaches to equivalence of expressions: (1) developing fluency within and among numeric, graphical, verbal, and symbolic representations; and (2) coordinating the symbolic features of CAS and paper-and-pencil techniques.

RESEARCH-INFORMED PERSPECTIVES AND ACTIVITY DESIGN

Arbaugh and colleagues (2010) discuss the importance of addressing issues that are valued by both teachers and researchers, including the goal of



supporting students' representational fluency with technology.

This article is based on the results of a research study that aimed to cultivate change in ninth-grade algebra students' representational fluency in solving linear equations using paper-and-pencil techniques and CAS (Fonger 2012). The research-informed perspectives and activities discussed here were tested in the first three weeks of a five-week collaborative teaching experiment between Ms. L. and the author. This experiment was the first experience Ms. L. had in teaching with the combined use of CAS and paper-and-pencil techniques.

For all class sessions, we had access to a classroom set of TI-Nspire CX™ CAS, but students did not have access to CAS at home. Ms. L. also had a projector to display the CAS teacher edition software as well as paper-and-pencil and CAS work from a document camera. The two activities discussed here require only slight modification for use with other handheld or computer-based CAS technologies.

Throughout the experiment, all activities were designed so that students would use both paper-and-pencil techniques and CAS to learn and do mathematics. An activity sequence also guided the design of tasks. We began with considering the equivalence of expressions from multiple representations; moved to looking at equations as equivalence relations that are sometimes, always, or never true; and closed with solving linear equations using multiple methods.

To support students' representational fluency, we prioritized the combined use of symbolic, verbal, numeric, and graphic representations to verify equivalence of expressions in the first component of this sequence. Throughout the activities, we wanted students to “use appropriate tools strategically,” an important mathematical practice (Common Core State Standards Initiative [CCSSI] 2010; Standards for Mathematical Practice 5, p. 6).

The framework of task-technique-theory, as discussed by Kieran and Drijvers (2006), highlighted relationships between key mathematical and technological ideas in each activity. The *task* is the activity that students engaged with; the *technique* is the means by which they accomplished that task; and the *theory* is the underlying mathematical idea that supported it. **Tables 1** and **2** give summaries of the lesson goals, tasks, techniques, and supporting theory for two activities. Activity 1 (see the **appendix**) gives a context for comparing the expressions $n + n + n + n + n + 1$, $2n + 2n + 1 + n$, and $4n + 2 + (n - 1)$. Activity 2 (see the **appendix**) requires students to reconcile paper-and-pencil analysis with tool-based results.

Table 1 Framework for Activity 1: Graphs and Tables to Verify Equivalence of Expressions

Lesson goals	• Use and reflect on paper-and-pencil and CAS representations of equivalent expressions • Justify the equivalence of expressions with graphs, numbers, and symbols
Tasks and techniques	• Translate symbolic expressions to a contextual situation (tile pattern) • Create graphs and tables from symbolic expressions using paper and pencil, CAS with a substitution operator, and function tables • Interpret the meaning of graphs and tables with respect to equivalence of symbolic expressions
Theory	• Different representation types can represent the same mathematical object from different yet complementary perspectives (Kaput 1989). • Multiple representations can take into account domain and range restrictions.

Table 2 Framework for Activity 2: Reconciling Paper-and-Pencil and CAS Symbolic Representations

Lesson goals	▪ Identify and explain patterns in symbolic expressions ▪ Verify the equivalence of expressions with symbols • Generalize the distributive property of multiplication over addition
Tasks and techniques	▪ Combine like terms by adding coefficients using paper-and-pencil techniques and CAS automatic simplification • Simplify expressions by applying the distributive property of multiplication over addition with paper-and-pencil techniques and the CAS expand() command • Reconcile differences between tool-based representations (if any)
Theory	▪ Equivalent expressions can be represented in multiple ways within one representation (e.g., symbolic). ▪ The equals sign can be used to indicate identity between equivalent expressions.

ENACTING TASKS

The tasks in activity 1 were designed to support students' representational fluency. Experience in creating and translating between graphs, tables, symbols, and words, along with the interpretation of multiple representations, were emphasized so that students could justify the equivalence of expressions from multiple perspectives.

One way to encourage the use of graphs and tables to verify equivalence of expressions is by using expressions that are “complicated enough” so that students cannot recognize equivalent symbolic forms simply by inspection. The expressions in activity 1 were chosen to reflect the structure of the tile pattern (see **fig. 1**). For instance, if n represents



Fig. 1 Counting edges as a function of the number of tiles leads to a variety of equivalent expressions.

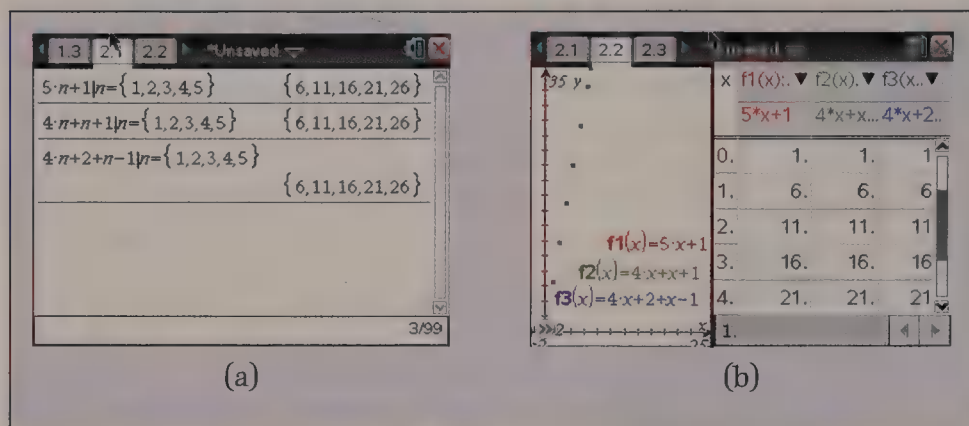


Fig. 2 The CAS operator | is used to generate the same numerical list for equivalent expressions (a). Split-screen graphs and tables show how color-coded equivalent expressions are “the same” (b).

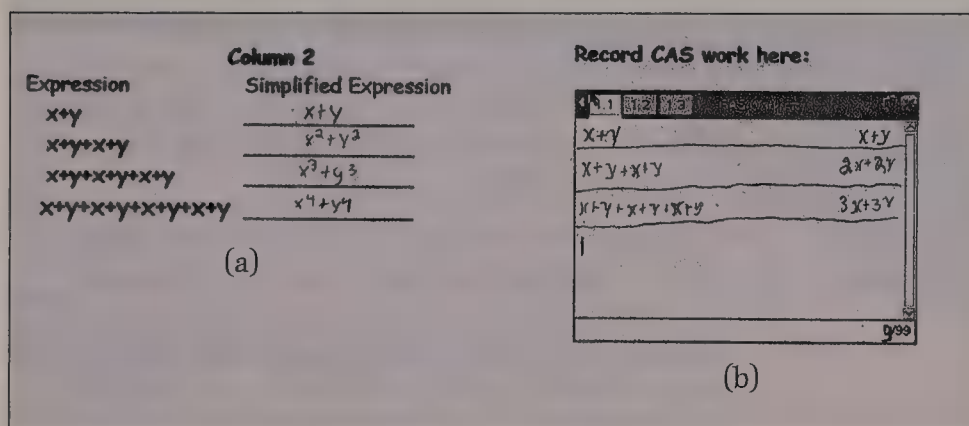


Fig. 3 This student's paper-and-pencil work (a) was not reconciled with his CAS work (b).

the number of tiles, the subtotal $2n + 2n$ represents the number of edges along the top and bottom of a row of n hexagons, with another n as the left-hand edge of each tile and one final edge on the rightmost tile. Regardless of which symbolic expressions are chosen, the context of tiles restricts the input and output to positive integers when values are graphed and viewed as tables. In the graphical representation, students might mistakenly draw a continuous line but could manipulate the properties of the line within a CAS Graph page to create a correct representation.

Before more formal terminology such as “equivalence” is used regularly in classroom discourse, students may first describe a graph and a numerical representation by saying that they appear to be “the same.” Students should be given opportunities to explain their interpretations and to respond to one another in discussion before more formal writing is expected.

Once the definition of *equivalence of expressions*

is introduced—equivalent expressions define the same pattern and can be shown by the exact same table, rule, or graph—students might be required to justify the equivalence of expressions using multiple types of representation. Students might also illustrate the definition of equivalent expressions with a meaningful example and screen shots or pictures and sketches of equivalent expressions. Some examples in multiple representational forms are shown in **figure 2**.

RECONCILING TECHNIQUES

High school students in a beginning algebra course have experience simplifying symbolic expressions from middle school, so graphical and numerical representations build the concept of algebraic equivalence in a meaningful way.

The design of activity 2, as that of activity 1, was inspired by research. First, CAS can be used as a tool to identify patterns and produce results in abstract symbolic forms (Heid and Edwards 2001), as in problems 1 and 5, respectively. Second, surprising CAS results should be negotiated with paper-and-pencil representations (Kieran and Saldanha 2008).

One way in which CAS can be used as a learning tool in studying equivalence of symbolic expressions is to follow Kieran and Saldanha's (2008) reconciling notion in which differences between CAS and paper-and-pencil forms are negotiated; the reconciled objects are reflected on; and a generalized relationship is proved. In the activity, the paper-and-pencil technique is used first, before CAS is used to complete the same task. Note that this activity structure is more than a “check”; specific attention is dedicated to negotiating differences between the tool-based representations. As shown in activity 2, problem 1, the paper-and-pencil technique involves combining like terms by adding the coefficients of the variables. The CAS technique used in problem 2—pressing **Enter** at the end of an entry line—is an automatic simplification giving students evidence that the input and output are equivalent expressions.

Teach Students How to Reconcile Tool-based Representations

Consider the student work in **figure 3a**, in which the paper-and-pencil simplified expressions in column 2 were not reconciled with the CAS output shown in **figure 3b**.

Reconciling results requires attention and reflection. If results from CAS and paper-and-pencil techniques are not intimately connected through specifically designed activities and classroom discussion is not focused to require reflection and reconciling activities, some opportunities to learn may be lost.

As a launch to activity 2, the teacher helped students with the specifically designed role-playing activity in which a sample task was completed and discussed. Ms. L. also explained the importance of students showing their work on their activity sheets; at this point in the activity, we were more interested in seeing their thinking than in seeing the correct answer. In other words, differences in students' paper-and-pencil work and their CAS work were expected, and we valued the careful recording of that activity so that we could pinpoint the use of CAS as a learning tool to support manual manipulation.

Draw Attention to Surprising Moments

The power in the approach advocated by Kieran and Saldanha (2008) is the notion of a “surprise element” between tool-based representations. When a surprise element occurs, students should be encouraged to take time to interpret their work. They can do so individually in a “fix” column, among small groups of students as they compare and negotiate differences in their work, and at the classroom level as the teacher guides the discourse to focus on the mathematical goal of the lesson. Activity 2, problem 3, was designed to address surprise elements.

One example of a surprise element in the classroom teaching experiment occurred when students used CAS to expand expressions, as in problem 3. The following exchange occurred after a student had made a prediction about the result of computing $-3 \cdot (2 \cdot x + 2)$ using paper and pencil and then executed `expand(-3*(2*x+2))` on his CAS.

Ms. L.: I want to know which one of these surprised you.

Gavin: Part a [the paper-and-pencil part].

Ms. L.: What did you write for the paper-and-pencil part?

Gavin: $3x + 12$.

Ms. L.: What did the CAS tell you?

Gavin: . . . $-6x - 6$.

Ms. L.: Yeah. OK. So this is a really good example. What they thought was going to appear did not appear . . . So what did you have to understand to fix it?

Gavin: That I have to multiply 3 all the way through to get the right answer.

Ms. L.: OK. So he said he had to multiply the $2x$ by -3 and he had to multiply the 2 by -3 to get the correct answer. Excellent job.

The focused questioning technique encouraged students to be explicit about the mathematics they were learning in this activity. Another summary technique might be to have students take time to reflect on and write about the changes they need to fix as a whole. Problem 4 was designed to focus

students' thinking before proving the generalized distributive property.

Move to the General Case

To begin thinking about a generalization of the distributive property, we used a “true” equation, as in problem 5. The equals sign between two equivalent expressions signifying an identity will yield a result of “true” on a CAS. One student explained that this is the case because “[they are] the same things but different expressions,” whereas another said, “They are the same [expression], just expanded and simplified.”

LESSON DESIGN PRINCIPLES

These two activities were introduced as a way to support representational fluency with equivalent expressions using paper-and-pencil and CAS techniques. One promising result of this research was the design and testing of lesson design principles that can be implemented across activities. We suggest that teachers—

- encourage both the creation of and interpretation of multiple representation types;
- make expressions “complicated enough” to warrant the use of graphs and tables to justify equivalence of symbolic expressions;
- emphasize prediction, action, and reflection when using CAS; and
- direct activities and classroom discourse toward reconciling differences in tool-based representations.

Whether students are expert CAS users or interested novices, it is my contention that they should have free access to available tools, with the expectation that they will use them judiciously. To cultivate the mathematical practice of using appropriate tools strategically, students need ample opportunities to harness the power of technology such as CAS and to learn CAS-specific techniques and how they are related to paper-and-pencil techniques.

Fostering a productive mathematical environment in which students are expected to be strategic users of paper and pencil, CAS, and multiple representations takes time. Ms. L. often modeled her own decision making on when to use CAS and how, yet students were also afforded opportunities to choose for themselves. In tasks like those in activities 1 and 2, which are specifically designed to use both tools, students need guidance to learn new practices such as reconciling. Role-playing activities and classroom discussions focused on the interplay between paper-and-pencil and CAS techniques may help. A task-technique-theory framework can also be used to help to elucidate the key aspects of

research-supported lesson design in a combined paper-and-pencil and CAS learning environment.

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APPENDIX: THE ACTIVITIES

Activity 1: Graphs and Tables to Justify Equivalence of Expressions

Once a tile design has been laid in place, the gaps are filled in with grout. Suppose that you are tiling a patio area with a single strip of regular hexagon tiles.



1. Your friend advised that it is best to grout completely around each tile. He came up with the following expressions for the total number of edges that need to be grouted, where n represents the number of hexagon tiles in the strip:

$$n + n + n + n + n + 1$$

$$2n + 2n + 1 + n$$

$$4n + 2 + (n - 1)$$

If the number of tiles is n , how does each expression relate to the total number of edges in the hexagon strip?

2. One way to justify that expressions are equivalent is to evaluate the expressions numerically and to create a graph to see whether you get the same thing. Do the first row by hand, before using your CAS.

Tool	Expression	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
Mental	$n + n + n + n + n + 1$					
CAS	$2n + 2n + 1 + n$					
Table	$4n + 2 + (n - 1)$					

3. Are the following expressions equivalent?

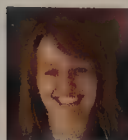
$$n + n + n + n + n + 1$$

$$2n + 2n + 1 + n$$

$$4n + 2 + (n - 1)$$

Use more than one representation to justify your answer.

Kieran, Carolyn, and Luis Saldanha. 2008. "Designing Tasks for the Codevelopment of Conceptual and Technical Knowledge in CAS Activity." In *Research on Technology and the Teaching and Learning of Mathematics: Vol. 2. Cases and Perspectives*, edited by Glendon W. Blume and M. Kathleen Heid, pp. 393–414. Charlotte, NC: Information Age Publishing.



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Activity 2: Equivalent Expressions with Paper-and-Pencil and CAS Techniques

1. Use paper and pencil to simplify the expressions below (combine like terms!).

Column 1		Column 2	
Expression	Simplified expression	Expression	Simplified expression
x		$x + y$	
$x + x$		$x + y + x + y$	
$x + x + x$		$x + y + x + y + x + y$	
$x + x + x + x$		$x + y + x + y + x + y + x + y$	

2. Use your CAS to help see the pattern in the simplified expression above in column 2. Add a **Calculator Page**. Type in each expression one at a time and press **Enter**.

3. Complete one row of the table at a time:

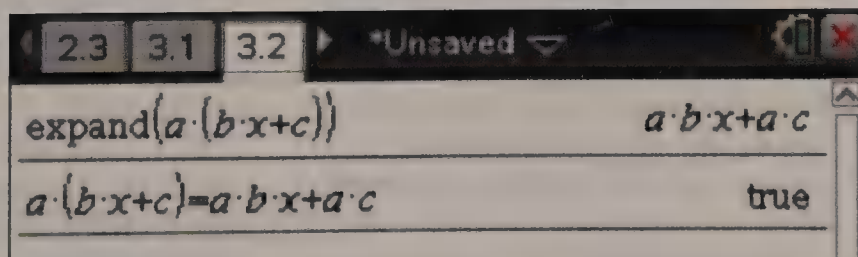
Original expression	Simplified expression (paper and pencil)	Simplified expression (CAS expand)	Explain and fix any differences
$-3 \cdot (2x+2)$			
$(3/2) \cdot (x-2)$			
$2 \cdot (3x-1)$			
$0.1 \cdot (5x+100)$			
$a \cdot (b \cdot x + c)$			

Talk to the students in your group. Be sure that your paper-and-pencil answer matches the CAS output!

4. Explain how the original expression relates to the expanded expression.

- (a) How is the coefficient determined?
- (b) How is the constant term determined?

5. Tom tested this rule on his CAS:



- (a) What do a , b , and c stand for?
- (b) What does the output "true" mean?

Predicting Future Gas Prices Using the Standards for Mathematical Practice

"How much do you think gas will cost when I graduate from high school?"

Like many commodities, the price of gasoline continues to rise, and these price changes are readily observed in gas stations' signage. Moreover, algebraic methods are well suited to model price change and answer the student's question. Over the course of one ninety-minute block or two forty-five-minute

classes, students build functions and interpret them in context. This article presents the activity, describes its implementation, provides sample student work, and discusses its relationship to the Standards for Mathematical Practice from the Common Core State Standards. Data used in the activity are available at <http://data.bls.gov/cgi-bin/survey/most?ap>.

FOCUSING ON THE STANDARDS FOR MATHEMATICAL PRACTICE

Engaging students in authentic problem solving has long been a focus for the teaching and learning of mathematics (Charles and Silver 1988; NCTM 1989). The Common Core State Standards encourage teachers to implement lessons that encompass the Standards for Mathematical Practice (SMPs) and have students "do mathematics" (Stein et al. 2010). Incorporating the eight SMPs (see **fig. 1**), this activity engages students in authentic problem solving and doing mathematics.

Several *Mathematics Teacher* articles provide background on regression analysis related to this activity (e.g., Edwards, Meagher, and Özgün-Koca 2008; Bossé 2005; Lesser 2004; Zelkowski and

Mayes 2008). This article focuses on the fluctuating commodity of gasoline prices over many years, but teachers may substitute a more stably priced commodity, such as bread or Red Delicious apples, which may be more appropriate for an introductory lesson. By incorporating the SMPS, students develop conceptual understanding of regressions and correlations not possible with traditional exercises emphasizing mechanics.

THE ACTIVITY

Since 2000, the price of gasoline has increased at a fairly steady pace, apart from one hiccup during the 2007–8 economic decline following the housing bubble burst. In this activity, students use a graphing calculator to visualize and analyze the data. Although written for the TI-Nspire™, the activity can be easily adapted for other models of graphing calculators.

The activity begins with a class discussion of the question, "How much do you think gas will cost when you graduate from high school?" Students identify the type of information they will need and brainstorm possible methods for answering the question. The initial discussion makes the problem more concrete and

Activities for Students appears six times each year in *Mathematics Teacher*, often providing in reproducible formats activity sheets that teachers can adapt for use in their own classroom. Manuscripts for the department should be submitted via <http://mt.msubmit.net>. For more information on the department and guidelines for submitting a manuscript, please visit <http://www.nctm.org/publications/content.aspx?id=10440#activities>.

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Standards for Mathematical Practice (SMPs)

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Fig. 1 The eight Standards for Mathematical Practice are part of the Common Core Standards for State Mathematics (CCSS 2010, pp. 6–8).

meaningful for students. Following the class discussion, the teacher presents students with data showing the average cost of a gallon of regular gas at the start of each calendar year since 2000. Working in small groups, students describe trends or changes in the price and record their group's consensus; respond to questions with mathematical reasoning; complete tasks using technology; and check their understanding by providing short answers that can be used as part of a formative assessment.

Next, students represent the data in multiple formats and interpret the data. Here, the teacher prompts students to think carefully about the graphical representation and to compare their new observations, based on the data and multiple representations, with their earlier observations (SMP 1). Students should be encouraged to experiment using different scales for each axis and think about how their choice of scale may influence their interpretation of the data. For example, displaying the x -axis from -1 to 13 and the y -axis from -1 to 4 provides a different view from the standard window settings (see **fig. 2**). From this perspective, the data appear to have much more variation and a more pronounced trend. By comparing these views, students better understand the quantities, the scale, and the units—all important to problem solving. During this phase of the activity, students should receive minimal guidance to encourage

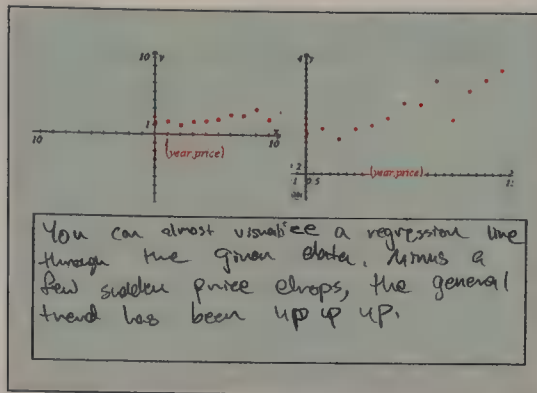


Fig. 2 An appropriate scale can show data trend and variation.

independent sense making (SMP 1).

The first set of “check for understanding” questions serves as formative assessment. Scanning responses to these questions, teachers will be able to assess students’ understanding of independent and dependent variables and their ability to read the data. The final question asks for students to predict when gas will be \$6.00 per gallon and to provide a rationale (SMP 2). It may be appropriate to pause for class discussion at this point in the activity. Teachers may ask students to predict the price at some point of interest in the future—for example, when the students graduate from college.

EXAMINING THE DATA GLOBALLY AND LOCALLY

Students next look for trends in the data. They should observe the increase in gas prices since 2000 and also notice that the price of gas oscillates in 2008 and 2009. Using the dynamic capabilities of the TI-Nspire, students can dynamically grab each outlier to better fit the trend of the data (see **fig. 3**). The technology allows students to reason abstractly about the trend while presenting their thinking concretely (SMP 5). For the remainder of the analysis, students should return to the original unmodified data.

For tasks 2 and 3, students break into small groups. The focus of each group’s discussion will be choosing which model to use to predict future gas prices. Students will also sketch a best-fit curve that they estimate by inspection, write an equation for their best-fit curve, and relate the equation to the trend. The underlying focus is on making sense of the mathematical structure of the data (SMP 7) while doing real-world math-

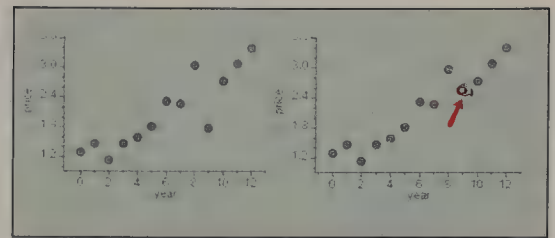


Fig. 3 Software allows for moving individual data points.

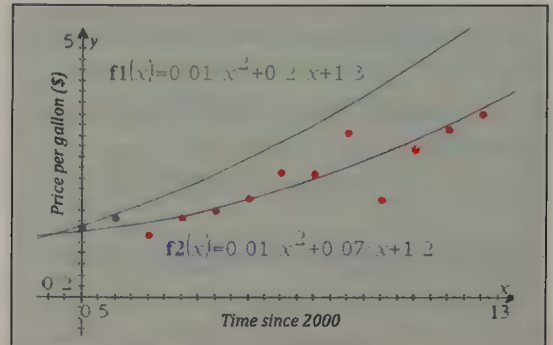


Fig. 4 Quadratic models have a leading coefficient close to zero.

ematical modeling (SMP 4).

Tasks 4 and 6 challenge students to develop two models—linear and quadratic. It is important that each group develop its own models. In this way, students must think about the structure of each model and make strategic modeling choices (Goldenberg 2000). Each group chooses a linear model that best fits the trend of the data in task 4 and chooses a quadratic model in task 6. The teacher may need to prompt some groups to examine small values for the coefficients when developing their quadratic model. Groups then discuss how the different values from their equations relate to gas prices. Students observe that the slope of their best-fit line represents the average increase in gas prices per year and that the y -intercept represents the price of gas at time 0 (the year 2000). For the quadratic model (see **fig. 4**), students observe that the leading coefficient is positive and close to zero, making the curve very wide and concave up; that the intercept again represents the price at time 0; and that the b term translates the parabola.

For tasks 5 and 7, the teacher provides students with the linear model $y = 0.172x + 1.076$ and the quadratic model $y = 0.006x^2 + 0.095x + 1.218$, respectively. To encourage students to compare models, the teacher may ask which model better estimates the price of gas on various dates, such as January 1, 2000, or January 1, 2011. Working

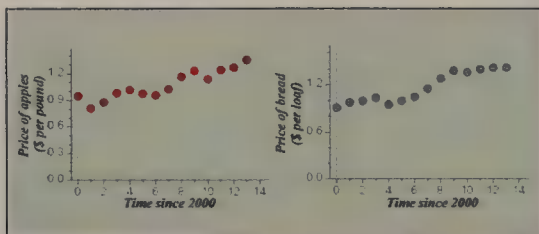


Fig. 5 Prices for different commodities may be analyzed.

together with graphs displayed on separate calculators facilitates comparison. Students now answer the original question—“How much will gas cost when I graduate?”—using each of the models.

CULMINATION OF THE ACTIVITY

This activity culminates with question 9, which ties together all the information and analysis that the students have been engaged with during this activity. This question extends students’ thinking beyond mechanical operations. It assesses students’ logical understanding and their ability to reason about functions. For example, students must attend to the precision of their answer and translate among units of time (SMP 6). Students must also extrapolate and justify their procedure, carefully examining their forecasts, both past and future (SMP 8). Many students choose the quadratic regression because of its interpolation accuracy.

EXTENSIONS

Given the variability of gasoline prices, teachers may elect to substitute a different commodity or include an additional one. The data for bread and apple prices can be found at <http://data.bls.gov/cgi-bin/surveymost?ap>. Price data for these commodities from 2000 to the present show much less scatter and an easily observable trend (see **fig. 5**).

The analysis can also be extended in several ways, some of which may be suggested by students. Some possibilities include extending the data set to monthly data, examining outliers, comparing residuals, and extrapolating with piecewise linear or exponential functions (see **fig. 6**). Students may also consider general price inflation by first converting the data to current dollars using an inflation calculator (available online at http://www.bls.gov/data/inflation_calculator.htm). Students may

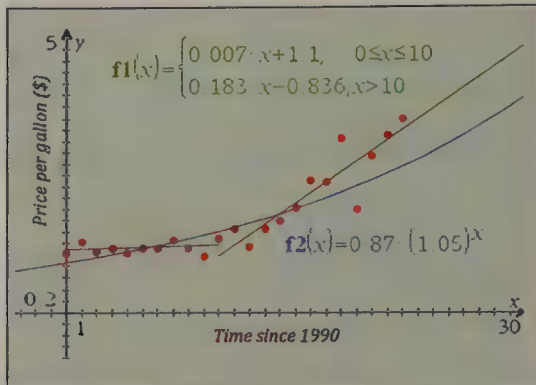


Fig. 6 Two options for extrapolating data are shown.

also use their analysis to debate public policies, such as the proper level of the minimum wage.

MODELING REAL-WORLD DATA

The idea of modeling real-world data is not unique. However, engaging students requires an intuitively appealing question. Moreover, developing the conceptual understanding necessary to use technology requires adhering to the eight Standards for Mathematical Practice. By combining a powerful initial question with effective pedagogy, this activity provides a strong foundation for students’ subsequent investigations. Future lessons in the same spirit may incorporate exponential or logistic functions. We encourage teachers to develop follow-up activities and make modeling a regular part of their curriculum.

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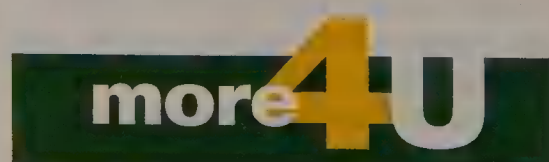


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This activity sheet is available to teachers as a Word document that can be copied and edited for classroom use; go to www.nctm.org/mt.

Activity 1: Predicting Future Gas Prices Using the Standards for Mathematical Practice

The price of gasoline in the United States has varied greatly in recent years. In this activity, you will explore relationships and trends in gasoline prices since 2000.

Question 1: Examine the table showing the price per gallon of regular unleaded gas averaged nationally at the start of the year. What do you notice about gas prices over the last 12 years? Consider different time intervals when answering the question.

Task 1: It sometimes helps to see a visual depiction of the mathematical relationships we are exploring. Enter the data from the table into your calculator and make a scatter plot of the data.

Question 2: According to the graph, how would you describe the trend? Does seeing the data graphically improve your understanding? Discuss the graph in your group.

Check for Understanding the Data

Check 1: What are the independent variable and the dependent variable for the data?

Check 2: For which year is the value of the independent variable zero?

Check 3: What was the average price per gallon of gasoline at the start of 2002?

Check 4: At the start of which year or years was the average price per gallon of gasoline \$1.79?

Years since 2000	Gas Price per Gallon
0	\$1.30
1	\$1.47
2	\$1.14
3	\$1.47
4	\$1.59
5	\$1.82
6	\$2.32
7	\$2.27
8	\$3.05
9	\$1.79
10	\$2.73
11	\$3.09
12	\$3.40

Early prediction: In which year do you think the price per gallon of gasoline will be \$6.00? Explain.

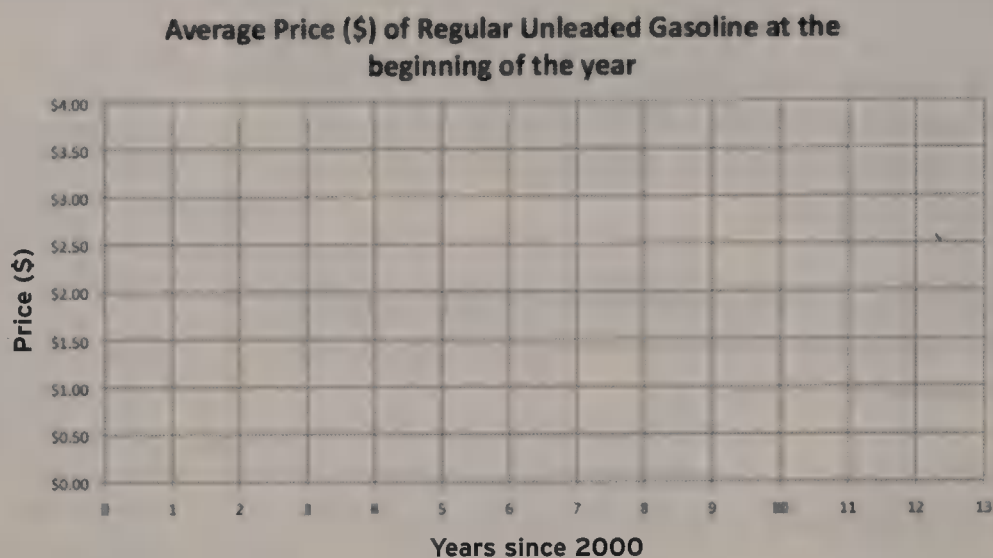
Question 3: Is there a period of time over which the price of gasoline remained fairly stable? If so, identify that time period and estimate the average price during that time. If not, explain why not.

Question 4: Identify a time period when gasoline prices fluctuated widely. Can you think of events during this time period that may have caused this fluctuation?

Task 2: Examine the graph of the data using the standard window settings. Does this scale represent the data well? Now edit the window values so that the view of the graph is more appropriate. How does your window compare with the one in the grid below?

Activity 1: Predicting Future Gas Prices Using the Standards for Mathematical Practice (continued)

Plot the graph on the grid below.



Task 3: Draw a straight line on the graph that best represents the trend in the data. This is called a *trend line*. Estimate and record the slope and y -intercept of your trend line. Enter the equation of your trend line on your calculator and display it on the scatter plot.

Check 5: What does the slope of your trend line represent? What are the units of slope for this problem? Record your responses.

Using Your Trend Line

Task 4: Record the equation for the trend line that you drew in task 3.

Question 5: (a) Within your group, compare each group member's trend line with yours. What similarities and differences do you notice? (b) As a group, choose one equation that best fits the data. Record the equation and explain why your group chose that equation.

Task 5: The linear regression equation can be found using your calculator. Record it here (round coefficients to the nearest thousandth).

Question 6: Compare the linear regression model with your group's trend line model from question 5. What similarities and differences do you see? Consider both the equations and the graphs.

Check for Understanding the Linear Model

Use the linear regression model to answer these questions. Show your work.

Check 6: What price does the model predict for January 1, 1995?

Check 7: What price does the model predict for July 1, 2005?

Check 8: At which time or times should the price of gasoline have been \$2.50?

Check 9: When do you expect gasoline to reach the price of \$6.00?

Activity 1: Predicting Future Gas Prices Using the Standards for Mathematical Practice (continued)

Examining the Data with a Quadratic Model

Task 6: Use your scatter plot to find a quadratic equation that fits the data well. Hint: Start with small values for the coefficients of x and x^2 . Record your equation.

Question 7: With your group, compare the equations that you chose. Choose one equation for your group and record it. Why did your group chose this model?

Task 7: The quadratic regression equation can be found using your calculator. Record it here (round coefficients to the nearest thousandth).

Question 8: Compare this model to your group's model from question 7. What similarities and differences do you see? Consider both the equations and the graphs.

Check for Understanding the Quadratic Model

Use the quadratic regression to determine the answer to these questions. Show your work.

Check 10: What price does the model predict for January 1, 1995?

Check 11: What price does the model predict for July 1, 2005?

Check 12: At which time or times should the price of gasoline have been \$2.50?

Check 13: When do you expect gasoline to reach the price of \$6.00?

Question 9: Compare your answers from checks 6–9 for the linear regression with your answers from checks 10–13 for the quadratic regression.

(a) How do the answers to these check questions compare?

(b) How do the check answers compare with the actual data set?

Question 10: What will be the price of gasoline when you graduate from high school? From college? Select the model that you believe best answers these questions. Think about the past and future when choosing your model. Defend your model choice and include a graph to support your choice.

EXTENSION

In the gas price activity, you examined trends that do not take inflation into account. In this extension, you will reproduce the analysis after adjusting the data for inflation.

Question 1: Do you think adjusting for inflation will change the data analysis you completed above? Explain your reasoning.

Task 1: Use the website http://www.bls.gov/data/inflation_calculator.htm to convert the price of gasoline to inflated prices for the current year. Add these values to the table.

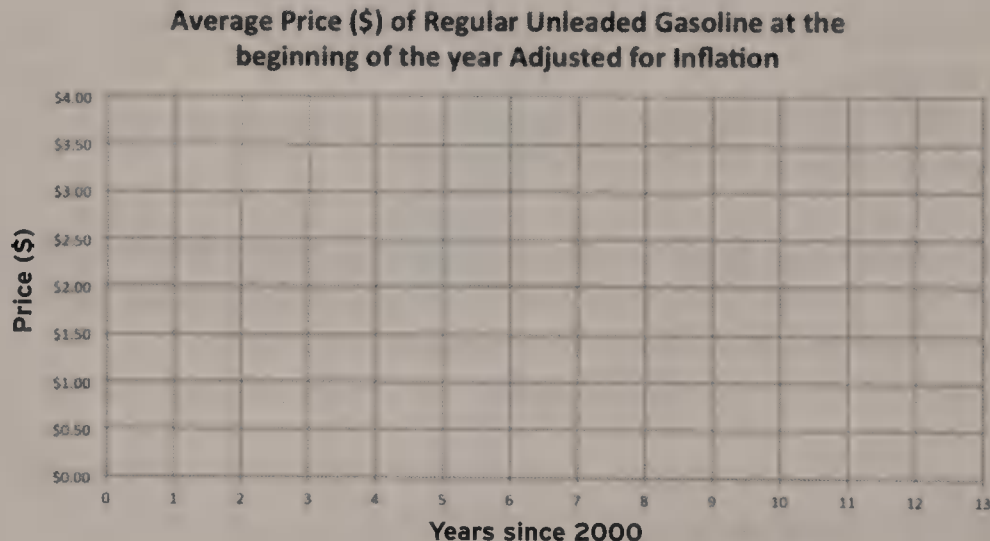
Activity 1: Predicting Future Gas Prices Using the Standards for Mathematical Practice (continued)

Question 2: Examine the table. How would you describe the trend in the inflation-adjusted gas prices? How do the adjusted prices compare with the nominal prices?

Years since 2000	Gas Price (nominal)	Gas Price Adjusted for Inflation
0	\$1.30	
1	\$1.47	
2	\$1.14	
3	\$1.47	
4	\$1.59	
5	\$1.82	
6	\$2.32	
7	\$2.27	
8	\$3.05	
9	\$1.79	
10	\$2.73	
11	\$3.09	
12	\$3.40	

Examining the Data with a Linear Model

Task 2: Create a scatter plot using the adjusted gasoline prices. Draw a trend line on the graph that you think best represents the trend in the data. Estimate and record the equation of your trend line.



Question 3: How well does your trend line fit the data?

Question 4: Do you believe that your linear model using inflation-adjusted prices would be good for predicting the price of gas when you graduate from high school?

Question 5: Do you believe that the linear model for inflation-adjusted prices will be better than the linear regression found using nominal prices (task 5 in the activity) for predicting future prices? Explain your reasoning.

Evaluating and Writing Dynamic Geometry Tasks

The advent of dynamic geometry software has changed the way students draw, construct, and measure by using virtual tools instead of or along with physical tools. Use of technology in general and of dynamic geometry in particular has gained traction in mathematics education, as evidenced in the Common Core State Standards for Mathematics (CCSSI 2010).

Research has shown the potential benefit of using technology, particularly

dynamic geometry tasks, to promote mathematical reasoning. The Common Core State Standards for Mathematics (CCSSM) high school geometry standards require students to “make formal geometric constructions with a variety of tools and methods” (p. 76), and dynamic geometry software is among the tools listed. Further, Hollebrands (2003) has noted that within such dynamic environments students have opportunities to consider invariant relationships through dragging as well as make corresponding conjectures and conclusions.

However, little guidance is provided to teachers for evaluating the quality of dynamic geometry tasks, much less for writing their own. The purpose of this article is to introduce a framework for analyzing and writing dynamic geometry tasks that are designed to engage students in mathematical reasoning. We begin by asking readers to compare two sample tasks, each of which is designed to engage students in developing and testing conjectures about parallelograms. We then introduce the framework and illustrate how it can be used to evaluate the potential of each task in accomplishing the desired result.

TWO TASKS TO COMPARE

Consider parallelogram tasks 1 and 2 (see **figs. 1** and **2**), where students are provided with a prepared sketch in the dynamic geometry environment. Quadrilateral $ABCD$ is constructed to be a parallelogram—that is, no matter what students do by dragging edges or vertices, $ABCD$ will always be a parallelogram. Each task therefore possesses *mathematical fidelity*, a guarantee that the objects in the sketch are what they are claimed to be. Without mathematical fidelity, the remainder of the task may become meaningless.

Read each task and consider this question: Which task contains prompts that encourage students to experiment and build their mathematical understanding? Which task would you consider using in the classroom?

A PROFESSIONAL DEVELOPMENT PROJECT

The impetus for this work originated in a professional development project in which the following research question was addressed: How do the design and use of tasks that incorporate dynamic

Connecting Research to Teaching appears in alternate issues of *Mathematics Teacher* and brings research insights and findings to the journal's readers. Manuscripts for the department should be submitted via <http://mt.msubmit.net>. For more information on the department and guidelines for submitting a manuscript, please visit <http://www.nctm.org/publications/content.aspx?id=10440#connecting>.

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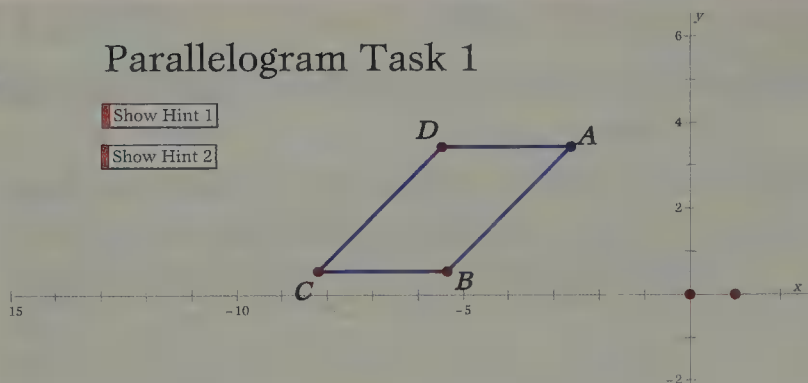
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Parallelogram Task 1

Show Hint 1

Show Hint 2



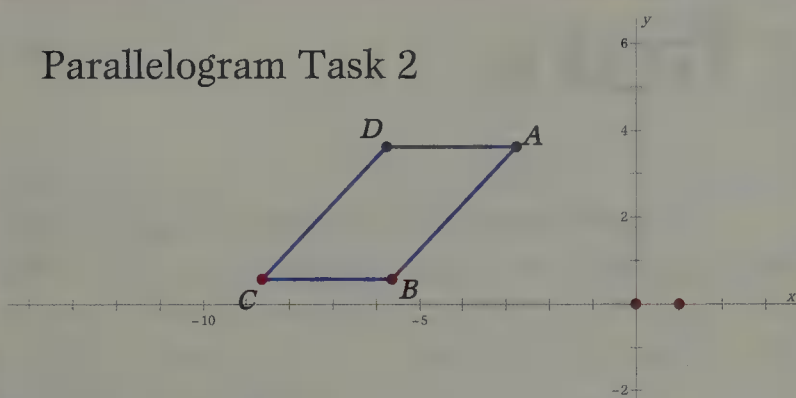
1. Describe what a parallelogram looks like.
2. Can you determine a relationship between the angle measures? Use hint 1 if needed. (Hint 1: Measure each angle.)

What do you notice about their measures?)

3. Can you determine a relationship between the lengths of the sides? Use hint 2 if needed. (Hint 2: Measure the length of each side. What do you notice about the lengths?)
4. Try dragging the vertices. Do your assumptions hold true?
5. Is it possible for all the sides to be the same length?
6. If yes, can you rename this figure?
7. Make one of your angles a right angle. What happens to the other angles of the parallelogram?
8. Construct diagonals. Mark the point of intersection and label it E . Measure AE , BE , CE , and DE . What do you notice?
9. What kind of relationship do the angles formed by the diagonals share?

Fig. 1 A teacher-written task is the basis for analysis.

Parallelogram Task 2



1. Drag parallelogram $ABCD$'s vertices. Write a conjecture about the relationship between the measures of opposite angles of this parallelogram ($\angle A$ and $\angle C$; $\angle B$ and $\angle D$).
2. Measure the four angles of parallelogram $ABCD$. Drag its vertices to make any size of parallelogram. Is your conjecture from question 1 true? Explain.
3. Drag parallelogram $ABCD$'s vertices. Write a conjecture about the relationship between the measures of opposite sides of this parallelogram (DA and CB ; CD and BA).
4. Measure the four side lengths of parallelogram $ABCD$. Drag its vertices to make any size of parallelogram. Is your conjecture from question 3 true? Explain.

5. Drag a vertex of parallelogram $ABCD$ until all four sides have the same measured length. Use **Construct Segment** to lock the lengths of all four sides of your parallelogram. What is the special name for a parallelogram with four sides of equal length?
6. Drag a vertex of the parallelogram until its angle is a right angle. What happens to the other three angles of the parallelogram? Why?
7. Construct diagonals. Mark the point of intersection and label it E . Drag vertices of parallelogram $ABCD$. Write a conjecture about the relationships among the line segments AE , BE , CE , and DE for this special type of parallelogram. (Hint: You named this type of parallelogram in question 5.)
8. Measure line segments AE , BE , CE , and DE . Drag the vertices of parallelogram $ABCD$ to make any size of parallelogram. Is your conjecture from question 7 true? Explain.
9. Drag the vertices of parallelogram $ABCD$. Write a conjecture about the relationships and types of angles formed by the diagonals ($\angle AEB$, $\angle BEC$, $\angle CED$, and $\angle DEA$).
10. Measure the angles formed by the diagonals ($\angle AEB$, $\angle BEC$, $\angle CED$, and $\angle DEA$). Drag vertices of parallelogram $ABCD$ to make any size of parallelogram. Is your conjecture from question 9 true? Explain.

Fig. 2 Technological actions are coordinated with mathematical prompts.

geometry software support teacher-facilitated mathematical discussions? The professional development project, which included twenty-four high school geometry teachers, had as an overarching goal to increase student success in high school geometry so that students would be able to pursue STEM-related college majors and careers. Part of the project involved teachers writing tasks for The Geometer's Sketchpad® software. The research team then analyzed video footage of classroom enactments using these tasks and attended to task characteristics and the emergent discourse.

Much like traditional geometry construction, many of these tasks required students to draw, measure, and construct in dynamic environments and respond to associated prompts. Unlike traditional construction tasks, some of these dynamic tasks required the use of dragging. We considered a high-quality dynamic geometry task to be one in which both the mathematical prompts and technological actions combine to focus students on understanding concepts, processes, and relationships as the sketch evolves.

The framework for dynamic geom-

etry task analysis presented here is based on the work of Sinclair (2003), who offers recommendations to promote student exploration with preconstructed dynamic sketches, and Smith and Stein (1998), who used Doyle's (1988) research on characteristics of academic tasks, and Resnick's (1987) work on high-level thinking skills to analyze hundreds of mathematical tasks. Smith and Stein defined four categories of cognitive demand associated with classroom tasks: memorization, procedures without connections, procedures with connections, and doing

Allowance for Mathematical Depth		Types of Technological Action	
Hierarchical Levels	Descriptions	Affordances	Descriptions
N/A	Prompt requires a technology task with no focus on mathematics.	N/A	Prompt requires no drawing, construction, measurement, or manipulation of current sketch.
0	Sketch does not have mathematical fidelity required to respond to prompt.	A	Prompt requires drawing within current sketch.
1	Prompt requires student to recall a mathematics fact, rule, formula, or definition.	B	Prompt requires measurement within current sketch.
2	Prompt requires student to report information from the construction. The student is not expected to provide an explanation.	C	Prompt requires construction within current sketch.
3	Prompt requires student to consider the mathematical concepts, processes, or relationships in the current sketch.	D	Prompt requires dragging or use of other dynamic aspects of the sketch.
4	Prompt requires student to explain the mathematical concepts, processes, or relationships in the current sketch.	E	Prompt requires the creation or consideration of multiple examples from which one can generalize.
5	Prompt requires student to go beyond the current construction and generalize mathematical concepts, processes, or relationships.	F	Prompt requires a manipulation of the sketch that allows for recognition of emergent invariant relationship(s) or pattern(s) among or within geometrical object(s).
		G	Prompt requires manipulation of the sketch that may surprise one exploring the relationships represented or cause one to refine thinking based on themes within the surprise. (Adapted from Sinclair [2003, p. 312])

Fig. 3 The framework for analysis includes two components: allowance for mathematical depth and types of technological action.

mathematics (p. 348). The first two categories include tasks that require a low level of cognitive demand; the third and fourth categories include tasks that require a higher level of cognitive demand. Note that the level of cognitive demand depends heavily on students' prior experiences, not just on how the task is phrased. What might be a routine task for one student could lead to an in-depth mathematical exploration for another.

A FRAMEWORK FOR TASK ANALYSIS

As we found in our analysis, tasks in which the technological actions were coordinated with the mathematical prompts seemed to have higher potential for engaging students in mathematical reasoning. Teachers and researchers need to evaluate such dynamic geometry tasks. To this end, we developed a framework that identifies characteristics of prompts associated with how the tasks are written.

The framework for analyzing dynamic geometry tasks (see **fig. 3**) is designed to identify qualities of dynamic geometry tasks that are best suited to support mathematical learning goals outlined in the CCSSM. The framework is divided into two components: allowance for mathematical depth and types of technological actions. These two divisions emerged after analyzing the teacher-written tasks. The levels of allowance for mathematical depth are based on the work of Smith and Stein (1998), whereas Sinclair's work (2003) provided a basis for the types of technological actions.

We defined a *mathematical prompt* as a written question or direction related to a sketch that requires a verbal or written response. Similarly, we defined a *technological prompt* as a question or direction that requires a drawing, construction, or measurement within or manipulation of a sketch. We coded levels of allowance for mathematical depth as numbers and types of technological actions as letters.

For example, prompt 7 from the first task in **figure 1** is both mathematical and technological and so was coded as (1, 2, 3, D).

The research team used the framework to code particular prompts of each parallelogram task from **figure 1**. Task 1 was written by a teacher beginning to use dynamic geometry software, and task 2 was written by the researcher developing the framework. Before reading further, we suggest that you take some time to consider how you would use the framework to code these prompts. Then think about what the coding implies about the potential of each task to engage students in developing mathematical understanding.

The first prompt, "Describe what a parallelogram looks like," requires the student to report information from the sketch with no need to perform a technological action and so was coded as (2, N/A). Prompt 4 for task 1, "Try dragging the vertices. Do your assumptions hold true?" received a code of

Table 1 Summary of Codes for Parallelogram Tasks 1 and 2

Prompt Number	Task 1 Codes	Task 2 Codes
1	(2, N/A)	(1, 2, 3, D)
2	(1, 2, 3, B)	(3, 4, 5, B, D, E, F)
3	(1, 2, 3, B)	(1, 2, 3, D)
4	(N/A, D)	(3, 4, 5, B, D, E, F)
5	(1, N/A)	(1, C, D)
6	(1, N/A)	(1, 2, 3, 4, D)
7	(1, 2, 3, D)	(1, 2, 3, D)
8	(1, 2, 3, B, C)	(3, 4, 5, B, D, E, F)
9	(3, N/A)	(1, 2, 3, D)
10		(3, 4, 5, B, D, E, F)

(N/A, D). The technological action of dragging led to the code D, but because the prompt in no way indicates whether the student's assumptions are about a parallelogram or even mathematical, the mathematical depth component was not applicable.

In task 2, prompt 2 built on prompt 1 as "Measure the four angles of parallelogram *ABCD*. Drag its vertices to make any size of parallelogram. Is your conjecture from question 1 true? Explain." The coding was (3, 4, 5, B, D, E, F). The student is required to measure (code B), drag (code D), consider multiple examples (code E), and recognize invariant relationships (code F). Consideration of the relationship between opposite angles (code 3) and explanation (code 4) allowed for mathematical depth. The coupling of explanation along with technological actions required the student to go beyond the current construction to address the relationship between opposite angles in any size parallelogram, and thus meets the highest level of allowance for mathematical depth for the answer (code 5).

Prompt 5 for task 2 reads "Drag a vertex of parallelogram *ABCD* until all four sides have the same measured length. Use **Construct Segment** to lock the lengths of all four sides of your parallelogram. What is the special name for a parallelogram with four sides of equal length?" The code (1, C, D) reflects requirements to recall a mathematics fact (code 1), drag (code D), and construct (code C). Prompt 5 did not

go beyond mathematical depth level 1 because the student was not required to report information from the sketch or explain why certain relationships were guaranteed.

Table 1 provides a summary of prompts and codes for each parallelogram task. Organizing the codes in this way provides a summary of the degree to which each task is written to promote mathematical reasoning. Notice that in task 1, five of the nine prompts are missing either mathematical or technological components; in addition, the allowance for mathematical depth never goes beyond level 3. In task 2, the prompts consistently include both a mathematical and technological component, and at times the allowance for mathematical depth reached level 5. We concluded that these qualities caused task 2 to have higher potential than task 1.

Each task begins with the same parallelogram and follows with similar but different prompts. Is one task more likely to engage students in the mathematical reasoning desired? Our hope is that, by comparing your analysis to ours, our methods for applying the framework will be clear. You will then be prepared to use the framework to analyze the quality of other dynamic geometry tasks and to write your own high-quality tasks.

HIGH-QUALITY DYNAMIC GEOMETRY TASKS

The dynamic geometry task analysis framework serves as a guide for analyzing and writing tasks for dynamic

geometry environments. High-quality tasks must contain prompts that require students "to go beyond the current construction and generalize mathematical concepts, processes, or relationships." Writing high-caliber prompts also requires the use of technological affordances, such as dragging, unique to a dynamic geometry environment. We found that certain coded combinations of allowance for mathematical depth and types of technological actions prompted students to be "doing mathematics," as Smith and Stein described (1998). For example, in task 2, a code of (3, 4, 5, B, D, E, F) resulted for a prompt that required the student to measure and then drag to consider relationships that go beyond parallelograms of one particular size. In responding to this prompt, the student is "doing mathematics" by exploring and understanding the nature of mathematical relationships. Prompts of this nature will encourage the student to make drawings, measurements, and constructions; explore relationships; and form deeper mathematical understandings.

Because the framework was based on tasks submitted by teachers who were beginning to use GSP in their geometry classrooms, a potential limiting feature is that it may best apply when evaluating tasks intended for novice users of dynamic geometry environments. As students continue to master technological actions and engage in cognitively demanding tasks, they may no longer need detailed directions that dictate technological actions. Performing such actions may become a usual expectation over time. Tasks that rank high on the framework may help establish these expectations. A fruitful area for further research may lie in exploring what expectations emerge as students continually engage in tasks that rank high on the dynamic geometry task analysis framework.

Implications for analyzing dynamic geometry tasks and writing high-quality tasks for dynamic geometry environments became apparent to us after examining many of these tasks in light of the discourse and understanding that emerged around their use. To organize our thinking and systematize

our categorization of task quality, we assessed tasks at the prompt level. In so doing, we crafted a framework that worked to assess task quality quickly on the basis of its prompts. To connect research to practice, we inferred that the framework would also prove beneficial to teachers seeking to analyze dynamic geometry tasks and write high-quality tasks for their students. To this end, we offer the dynamic geometry task analysis framework as a guide for teachers desiring to harness the full potential of dynamic geometry environments for the betterment of student conceptual understanding.

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Connecting Research to Teaching *Bridging the Divide*

Have you ever noticed a gap between research and practice?

How can research effect change in the classroom?

The Connecting Research to Teaching department of *MT* invites classroom teachers to explore research findings in relation to their practice. We also invite education researchers to demonstrate how results from their studies shape classroom practice. Findings from collaborative action research projects are also encouraged. Evidence of connections from research to practice commonly includes student work and brief transcripts from interviews or classroom videos.

Some ideas to consider:

- Does your research have the potential to change classroom practice?
- What urgent questions emerge in learning environments?
- In what ways is current research reflected in the classroom?
- How does research inspire innovation in the classroom?

Articles published in this department are eligible for NCTM's Linking Research and Practice Outstanding Publication Award. For more about this award, go to <http://www.nctm.org/news/content.aspx?id=31248>.

Manuscripts should include the research context and should be written with the classroom teacher as reader in mind. A good way to focus your material is to identify one clear message for teachers. Sharing student work where appropriate is encouraged. Manuscripts must be 2500 words or fewer.

See <http://www.nctm.org/news/content.aspx?id=22602> for submission guidelines. If you have questions about whether a particular idea would be a good fit for the department, contact Margaret Kinzel, mkinzel@boisestate.edu, or Laurie Cavey, lauriecavey@boisestate.edu.



NATIONAL COUNCIL OF
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CALL FOR MANUSCRIPTS

High-Leverage iPad Apps for the Mathematics Classroom

The use of iPads® in the classroom is growing. In the 2013–14 school year, 57 percent of schools planned to invest in iPads (Netop 2013). This investment can benefit mathematics classrooms only if teachers know which apps they can use to help students develop deeper mathematical understanding. Although learning about and developing facility with various apps is valuable for mathematics teachers, the process can be difficult, overwhelming, and time-consuming. To get started, we recommend one app, Dropbox, that can be used to share materials within the classroom setting, and then we suggest three free, easy-to-use mathematics apps: Sketchpad Explorer, Data Analysis, and MathGraph (see the **table** on p. 711).

USING DROPBOX TO SHARE FILES WITH IPADS

Because traditional methods of sharing materials (i.e., a USB stick) are not possible on iPads, teachers need easy ways to share documents and files with students. In addition to posting files on class websites (e.g., Edmodo®), Dropbox is a convenient way to share classroom materials. Dropbox users can store files on the Internet and access these materi-

als from any computer or iPad. Materials can be shared through Dropbox in two different ways; however, both methods require the teacher to have a Dropbox account and classroom iPads to be connected to a Wi-Fi network.

Possibly the simplest way to share files using Dropbox is to set up a class folder. This option gives students access to any files saved in the folder. To set up a class folder, the teacher must log in to his or her online Dropbox account at www.dropbox.com.

Method 1: Sharing with Students without Dropbox Accounts

To share a file or folder with a class, select it and then click the chain-link icon in the far-right corner (see **fig. 1**). To share by entering e-mail addresses, enter students' e-mail addresses, and the link will be sent to them. To share by using a URL, choose **Get link** on a computer to automatically copy the URL to the clipboard. On an iPad, users must manually copy the URL (see **fig. 2**). Now the link can be pasted on a social media site or in an electronic document. Once students have the link or have received the link by e-mail, they have access to the linked file or folder. If teachers share the link for a file or folder,

Technology Tips, which provides a forum for innovative uses of technology in the teaching and learning of mathematics, appears seven times each year in *Mathematics Teacher*. Manuscripts for the department should be submitted via <http://mt.msubmit.net>. For more background information on the department and guidelines for submission, visit <http://www.nctm.org/publications/content.aspx?id=10440#tech>.

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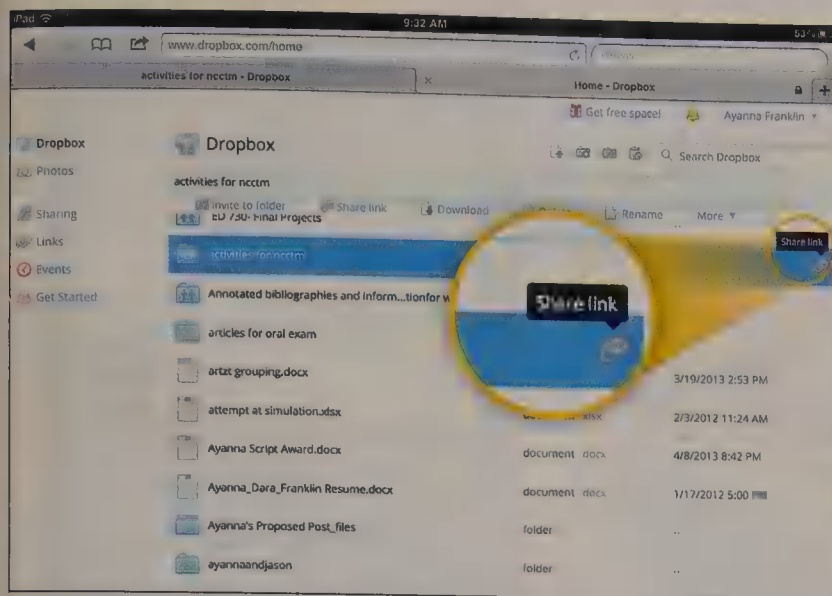


Fig. 1 Click on the chain-link icon on the folder to give access to the electronic document.

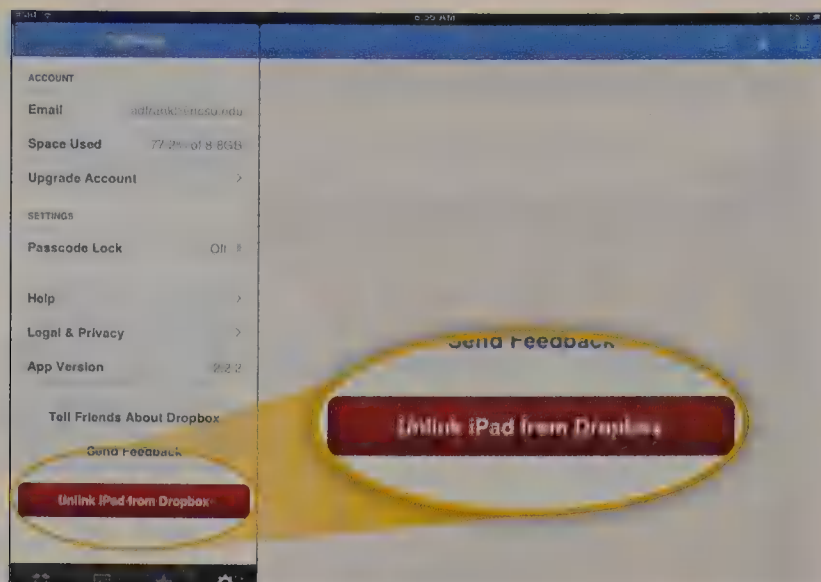


Fig. 3 This image shows the Dropbox logout screen.

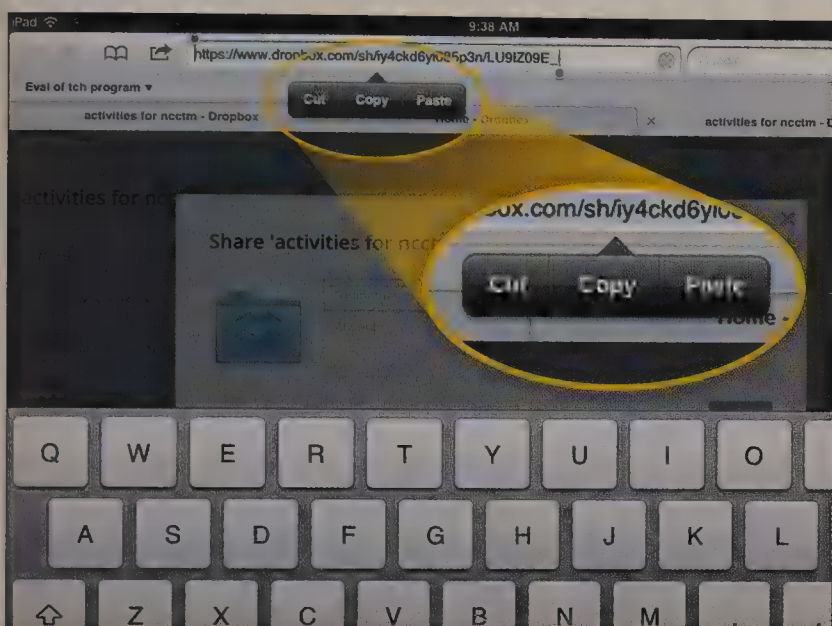


Fig. 2 Copy the URL to share the folder or file with students.

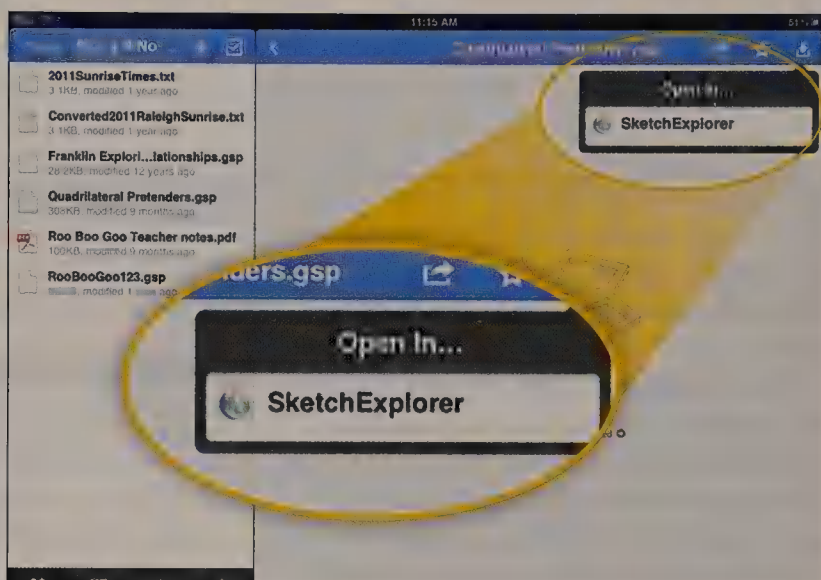


Fig. 4 Apps available to open the current file type show when users click on the download icon.

students can download files, but they cannot modify files or add any documents to the folder. Whenever students need to access folder contents, the link will need to be used (see **fig. 1**).

Method 2: Sharing with Students with Dropbox Accounts

If students have individual Dropbox accounts, they can be invited to the class folder. To invite them, choose the folder and then select the invite-to-folder icon (with the rainbow) at the top of the screen. Enter students' e-mail addresses in the box. Students can accept the invitation through their e-mail and access folder contents once they are logged into Dropbox. Once students have been invited to a class folder, they can access any files in that folder as well as add or remove files from the folder.

To log in to individual Dropbox accounts, students should click on the dropbox icon and enter their e-mail address and password. Doing so will link their Dropbox account to that particular iPad. Because most iPads in schools are shared by many students, students need to unlink their accounts from the iPad before returning it by choosing the settings icon (which looks like a gear) and then the **Unlink iPad from Dropbox** button (see **fig. 3**).

Finally, once students have accessed a file in Dropbox, they must choose a program to open it. By clicking on the download icon at the top right of the document, students will see a list of the available apps that can open the current file type. **Figure 4** shows the list of available apps to open a Geometer's Sketchpad® file and a data file.

THE SKETCHPAD EXPLORER APP

Dropbox can be used to share The Geometer's Sketchpad (GSP) files. GSP is one of the most popular applications for creating and interacting with dynamic geometry content. Sketches can be created to support either algebraic or geometric contexts. Sketchpad® Explorer is a free app that allows users to interact with all files created in GSP version 5 and most files created in GSP version 4. The main difference between GSP and Sketchpad Explorer on the iPad is that only GSP contains the ability to construct new objects, create new measurements, and perform calculations. **Figure 5** shows the sketch Superpose Triangles (Blackhawkmath 2013) in GSP and Sketchpad Explorer. Notice the missing left-hand toolbox and lack of menu access in Sketchpad Explorer.

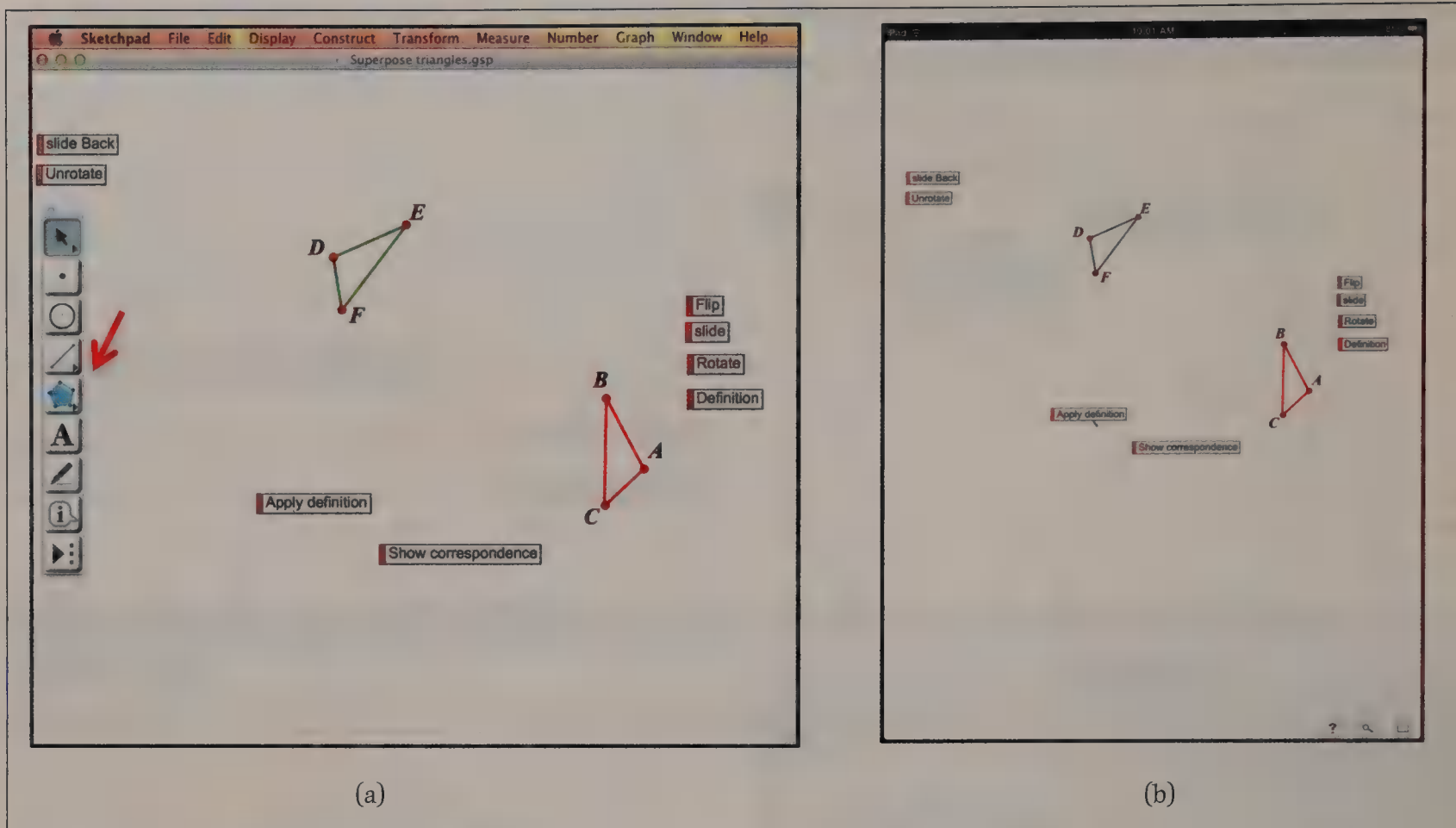


Fig. 5 The Superpose Triangles sketch is displayed in GSP (a) and in Sketchpad Explorer (b).

To interact with files created in GSP on the iPad, users need everything to have been prebuilt, premeasured, and precomputed in the form of sliders and buttons. Without these buttons, students would not be able to perform these actions on the iPad. For example, the sketch in **figure 5** contains buttons that allow students to flip, rotate, and slide triangles. This sketch uses rigid transformations to help students think about triangle congruency. For guidelines on creating iPad sketches, visit <http://sketchexchange.keypress.com/forum/showthread.php?45-Guidelines-for-iPad-Sketch-Design>.

Rather than constructing, Sketchpad Explorer allows students to use action buttons and their fingers, through multitouch capabilities, to explore sketches. Multitouch allows the use of more than one finger to drag and manipulate multiple figures simultaneously. Students can zoom in by pinching the screen or scroll by dragging two fingers on the screen. For detailed instructions on how to use Sketchpad Explorer, visit http://www.dynamicgeometry.com/General_Resources/Sketchpad_Explorer_for_iPad.html.

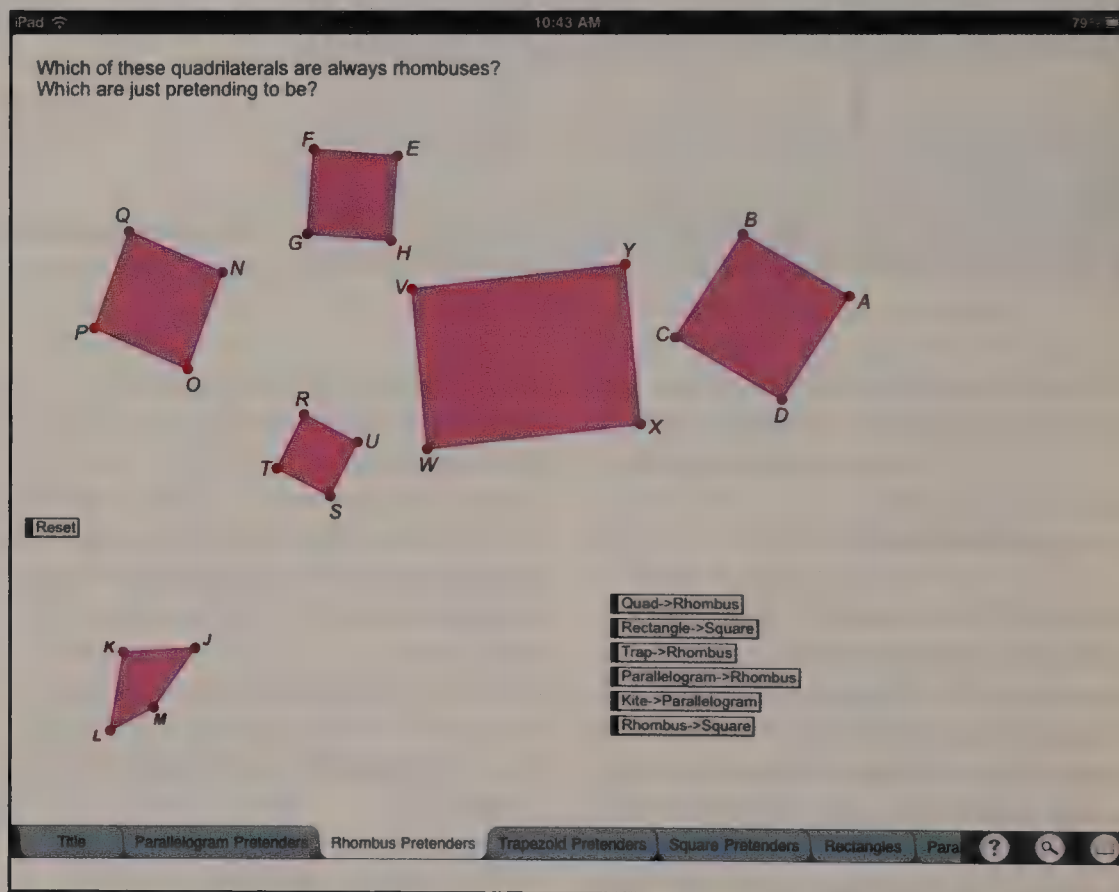


Fig. 6 Quadrilateral Pretenders Activity is a GSP sketch with premade buttons.

When Sketchpad Explorer is installed on an iPad, it contains a library of sample documents, which can be accessed using the open book icon. This icon also gives users access to a wide

variety of free downloadable files from the Sketch Exchange website (<http://sketchexchange.keypress.com/ipad>). This website contains sketches that can be searched using key words. Complete

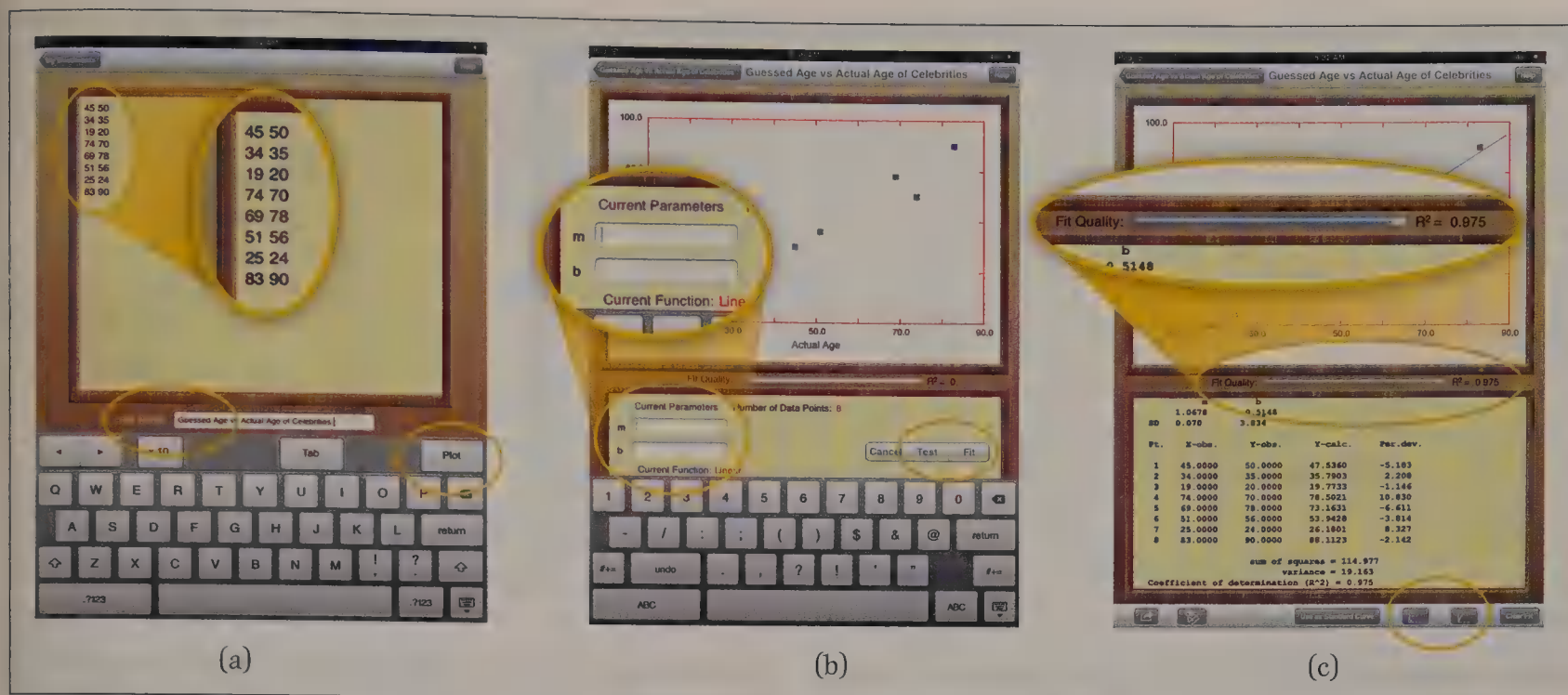


Fig. 7 Enter the guessed age and actual age data into the new document (a). Students can test parameters to fit the data (b) and graph the regression line (c).

activities can also be downloaded at the Sketchpad Learning Center (<http://learningcenter.dynamicgeometry.com/SampleActivities.htm>). One activity currently available is Quadrilateral Pretenders (see **fig. 6**). This activity allows students to investigate quadrilaterals' properties by dragging vertices and edges. This file could help students review properties of quadrilaterals and develop an understanding of the hierarchical structure of quadrilaterals.

THE DATA ANALYSIS APP

The Data Analysis App allows users to plot and analyze bivariate data. This free app quickly produces best-fit curves and their equations. Lists of two-variable data can be imported from .txt, .csv, and csv.gz files or entered directly into a list. A wide range of fitted curves can be applied to data, including the following: linear, quadratic, and cubic polynomials; and power, exponential, and log functions. The axes, labels, symbols, and color can be modified on graphs.

Using Data Analysis, students can analyze many different bivariate data sets. To illustrate the capabilities of this app, ask students the following question: "How good are you at guessing celebrities' ages?" To explore this question, present students with a list of celebrities' ages. Ask students to guess the celebrity ages. Then present the actual ages. Now

students have bivariate data (actual age, guessed age) to analyze.

To enter the data into Data Analysis, students should choose the **New Document** icon in the upper left-hand corner of the screen. Now students can enter each ordered pair on a single line separated by a single space (see **fig. 7a**). Once each ordered pair has been entered, students can name the file and choose **Plot**. Doing so shows the data points in a scatter plot. To customize scatter plots, choose the graph customizer button (which looks like a scatter plot of points) (see **fig. 7c**). Students should label the x -axis Actual Ages and the y -axis Guesses and add a title for the graph. They can also be asked to explain why the axes are labeled as such.

Once the data have been entered, functions can be overlaid on the scatter plot to graphically model the data. The default function (Current Function) is linear, which appears to be appropriate given the linear trend of the guessed age data. To model other data, press the other model button (which looks like a $\sqrt{x}$ graph) (see **fig. 7c**) to access alternatives.

Students can model their data using the **Test** button at the bottom of the screen. They can make educated guesses about the slope and y -intercept for the best-fit line for their data (see **fig. 7b**). In this case, the best-fit line tells students how good a guesser they are. After entering the estimations, users must press

Test to see the line. Students can repeat this process to test better estimates. This process helps reinforce how changing coefficients of a linear equation changes the graphical representation. To compute the least squares regression line, students should press **Fit** (see **fig. 7b**). Doing so also provides values for the slope, y -intercept, R^2 , and other associated least-squares data. The app also visually shows R^2 as a bar in the middle of the screen titled Fit Quality (see **fig. 7c**).

THE MATHGRAPH APP

The MathGraph App is an interactive app that can be used to graph various polynomials, absolute value functions, ellipses, exponential functions, and logarithmic functions. This free app allows users to modify function coefficients by using sliders or by entering numerical values between -10 and 10 . MathGraph also defines functions and function coefficients to connect equation and graphical representations. It allows students to make graphs quickly and visualize translations, dilations, and function inverses.

MathGraph has two distinct modes, Show Me and Quiz Me, that support graphical explorations. The default on opening the app is the Show Me mode. To illustrate the capabilities of this app as an inquiry tool, we will use Show Me to consider the following question: How do increases or decreases in the parameters

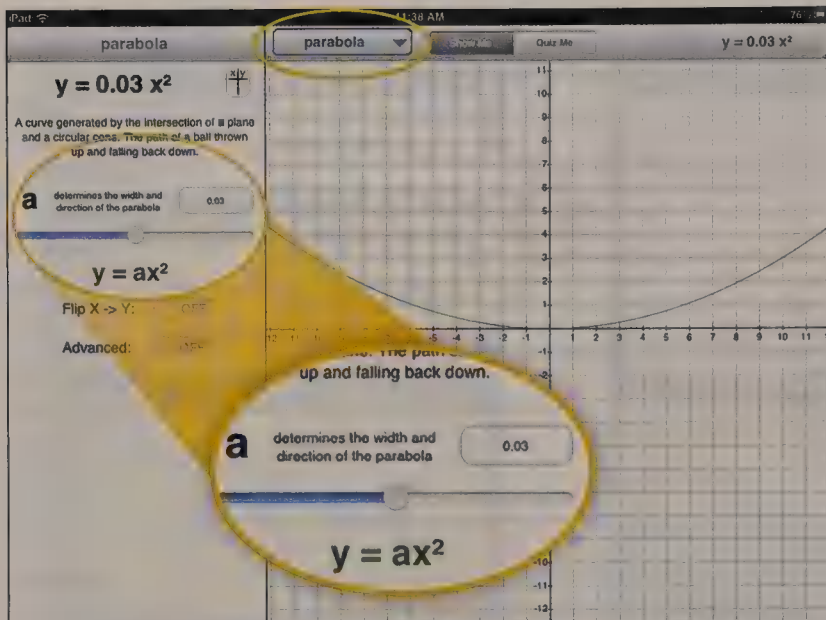


Fig. 8 Set the value for the leading coefficient using a slider.

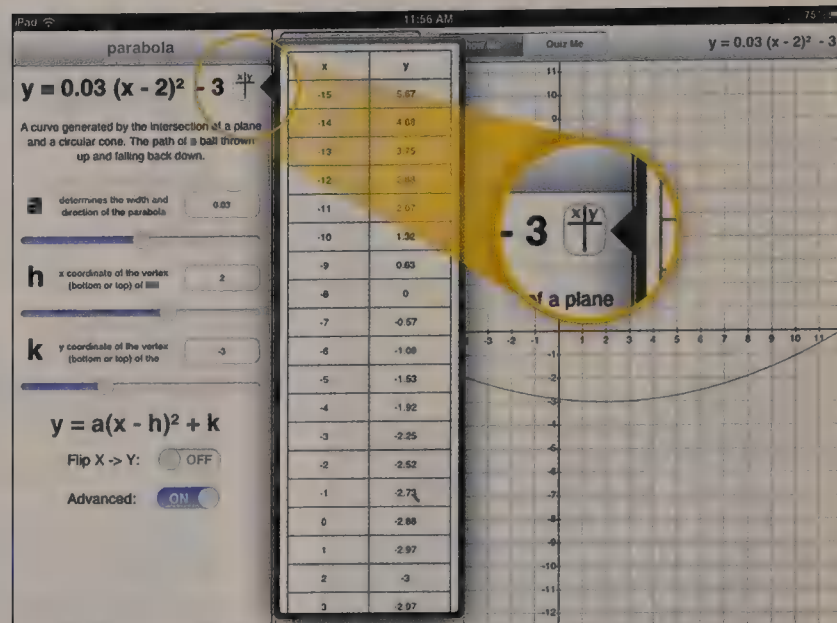


Fig. 10 Select the icon for the table of x,y-values to see the coordinate pairs.

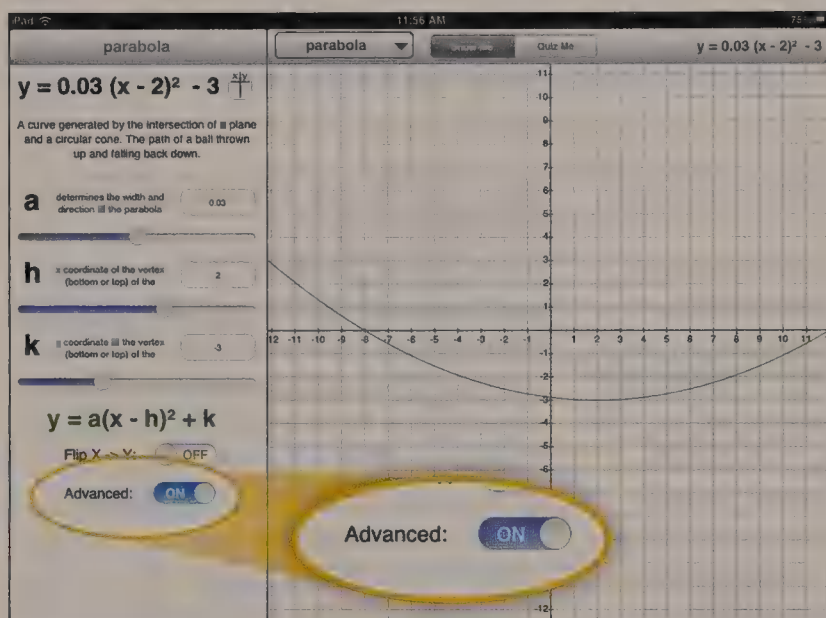


Fig. 9 Turn on the Advanced option to set h and k .

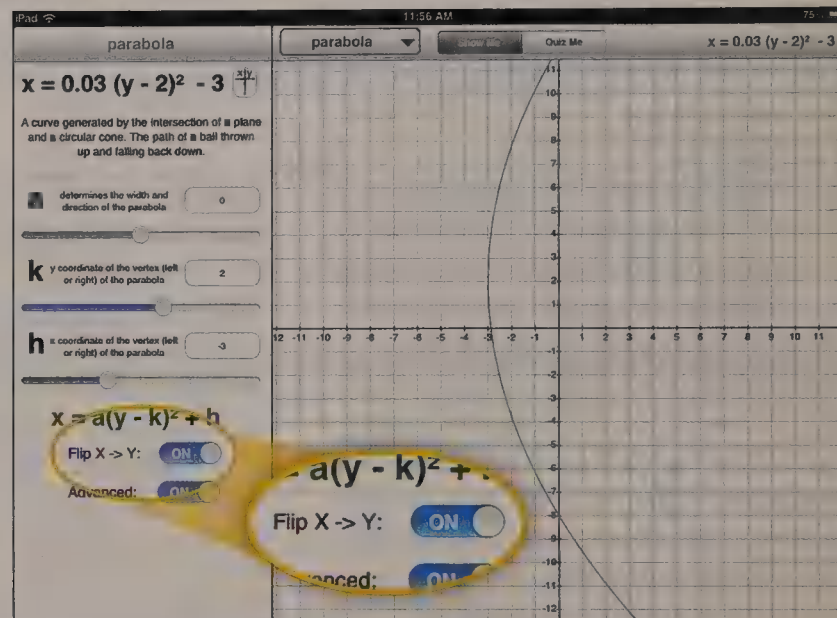


Fig. 11 Select Flip X -> Y to see the inverse function.

a , h , and k translate a parabola? A sample lab using this app can be found at <https://sites.google.com/site/mathgraphipadapp/>. Be sure to position the iPad horizontally to see the coordinate plane and the parameter sliders simultaneously.

To explore the parameters of parabola, choose **parabola** from the drop-down menu. The first parameter visible is a , the leading coefficient (see fig. 8). Have students predict how changes in a will affect the graph's shape. Then have them test their predictions by using the slider controlling a or by entering specific numerical values. Parameters can take on decimal values to the hundredth place between -10 and 10 . To enter specific numerical values, click on the value displayed and type in the new value using the pop-up calculator. Next, students should turn on the advanced option to

see parameters h and k , the vertex coordinates, and their definitions (see fig. 9). Again have students predict graphical changes and test their hypotheses.

To assist students with connecting function representations, the app includes the symbolic, tabular, and graphical forms of each function. To explore how coordinate values are affected by changes in parameter values, students can click on the icon for the table of the x,y -values (see fig. 10).

As an extension activity, ask students, "What do you think will happen if the x,y -variables of the parabola are flipped? Will the parameters create the same movements?" Once students express their hypotheses, have them turn on the **Flip X -> Y** option to test their hypotheses (see fig. 11). Once this activity is completed with parabolas, students

can use the same questions to explore polynomials (curves) of higher degrees.

AN IPAD IS A POWERFUL TOOL

As these apps show, the iPad can be a powerful tool to supplement mathematics instruction for a variety of mathematics content. Sketch Explorer provides students access to a dynamic geometry environment that allows them to test conjectures. Data Analysis allows students to import, model, and analyze data. Finally, MathGraph allows students to interact with transformations of functions. These three free apps can help teachers reconceptualize ways to teach common topics in their classrooms.

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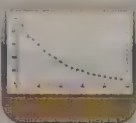


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leagues at North Carolina State University in Raleigh. They are interested in teachers' applications of technology to conceptual learning in mathematics classrooms. **HOLLYLYNNE S. LEE,** hollylynne@ncsu.edu, is a professor at North Carolina State University. She is interested in the use of technology for the teaching and learning of mathematics.

Table Icons, Screenshots, and URLs

App	Icons	iTunes URL
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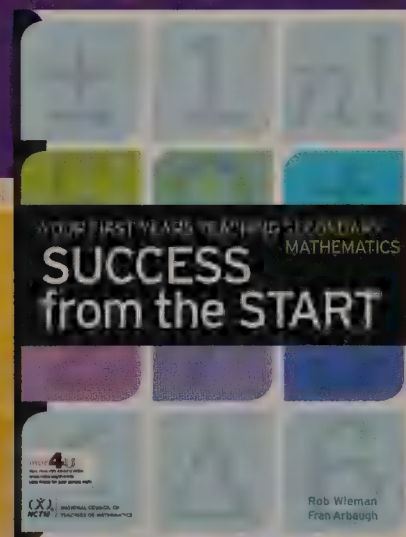
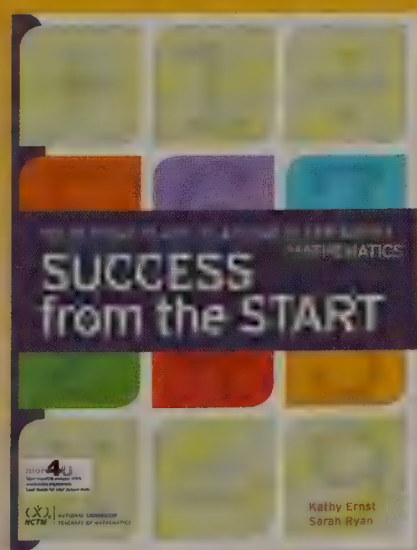
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Twists on the Tower of Hanoi

At a party that I attended, the hosts gave their guests the Tower of Hanoi puzzle with alternating dark and light discs and a challenge to move the 7 discs to a new post. (I disqualified myself because I knew how to solve the challenge.) However, the hosts' son and daughter-in-law misunderstood the directions and moved the dark discs to one side post and the light discs to the other side post. I immediately wondered, "How many moves did they take, assuming that they made the most efficient moves? How can their interpretation of the problem be generalized to n discs?"

The standard game can be played on the Internet or through any number of free smartphone applications. Those familiar with the Tower of Hanoi know that to move n discs to another post requires $2^n - 1$ moves, so the standard puzzle would take $2^7 - 1$ moves. Let's begin to investigate a modified game in which the only requirement is that the

two colors be separated. The task for 7 discs can be achieved as outlined in **figure 1**.

A TWO-COLOR CHALLENGE

We can now develop a general strategy involving n discs for some large value of n . Notice that we can assume that the bottom white disc is in its correct position. To compute the number of moves needed to separate the two colors, we follow these steps:

1. To uncover the largest black disc, move the top $n - 2$ discs to the left post using $2^{n-2} - 1$ moves.
2. To place the largest black disc in its correct position, move it from the center post to the vacant right post in 1 move. Now two discs are in the proper position; one white disc is alone at the center post, and one black disc is alone at the right post. A stack of $n - 2$ alternately colored discs is waiting at the left post. We have used $2^{n-2} - 1 + 1 = 2^{n-2}$ moves to place the second disc in its proper position.
3. To uncover the next white disc, move the top $n - 3$ alternately colored discs from the left post to the right post using $2^{n-3} - 1$ moves. (Consequently, the resulting stack of $n - 3$ has its bottom disc in the correct position.)
4. Place this white disc in its final position, moving it from the left to center post in 1 move. In steps 3 and 4, we have used $2^{n-3} - 1 + 1 = 2^{n-3}$ moves to place the third disc in its correct position. Notice that the bottom disc of the resulting stacked $(n - 3)$ alternately colored discs is already in the proper position.

Delving Deeper offers a forum for classroom teachers to share the mathematics from their own work with the journal's readership; it appears in every issue of *Mathematics Teacher*. Manuscripts for the department should be submitted via <http://mt.msubmit.net>. For more background information on the department and guidelines for submitting a manuscript, visit <http://www.nctm.org/publications/content.aspx?id=10440#delving>.

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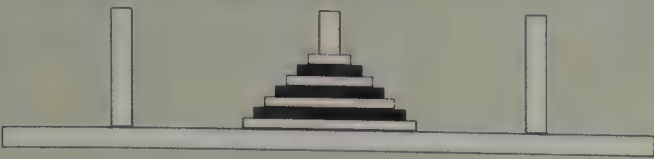
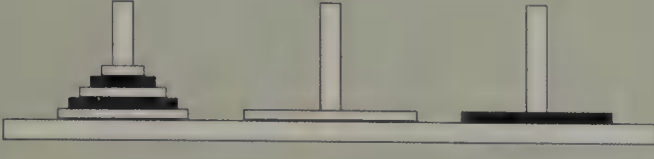
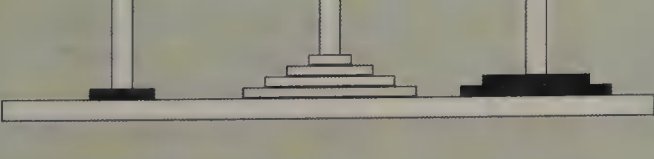
	The initial position shows the seven discs on the center post. The bottom white disc is in its correct position.
	1. Uncover the largest black disc by moving the top five discs of the stack to the left post ($2^5 - 1$ moves).
	2. Sort the largest black disc to its correct position by moving it from the center post to right post (1 move). Two discs are now correctly placed.
	3. Uncover the white disc on the left post by moving the top four discs of the stack to the right post ($2^4 - 1$ moves).
	4. Sort the white disc from the left to center post (1 move). Four discs are now correctly placed.
	5. Uncover the next white disc by moving the top two discs from the right post to the left post ($2^2 - 1$ moves).
	6. Sort the white disc from the right post to the center post (1 move).
	7. Sort the white disc from the left post to the center post (1 move).
	8. Sort the black disc from the left post to the right post (1 move). All discs are now correctly placed.

Fig. 1 In a modified game, the seven discs are separated by color.

Table 1 Results Expressed as a Sum of Powers of 2

Number of Discs in Original Tower	Number of Moves to Separate Colors	Powers Not Represented in the Sum
1	0	
2	2^0	
3	$2^1 + 2^0$	
4	$2^2 + 2^1$	2^0
5	$2^3 + 2^2 + 2^0$	2^1
6	$2^4 + 2^3 + 2^1 + 2^0$	2^2
7	$2^5 + 2^4 + 2^2 + 2^1$	$2^3, 2^0$
8	$2^6 + 2^5 + 2^3 + 2^2 + 2^0$	$2^4, 2^1$
9	$2^7 + 2^6 + 2^4 + 2^3 + 2^1 + 2^0$	$2^5, 2^2$
10	$2^8 + 2^7 + 2^5 + 2^4 + 2^2 + 2^1$	$2^6, 2^3, 2^0$
11	$2^9 + 2^8 + 2^6 + 2^5 + 2^3 + 2^2 + 2^0$	$2^7, 2^4, 2^1$
n	$2^{n-2} + 2^{n-3} + 2^{n-5} + 2^{n-6} + 2^{n-8} + \dots$	$2^{n-4}, 2^{n-7}, 2^{n-10}, \dots, 2^{n-(3k+1)}$ $k \in \{\text{positive integers}\},$ $n - (3k + 1) \geq 0,$ so $k \leq (n - 1)/3$

These steps establish an algorithm that will reduce a stack of (n) alternately colored discs to a stack of $(n - 3)$ alternately colored discs. In the process, three discs are correctly sorted to their proper post. We can visualize moving a stack to uncover a lower disc and then placing that disc in its correct position. The algorithm begins with (n) alternately colored discs, uncovers, and sorts by moving stacks of $(n - 2)$ and $(n - 3)$ discs, thus reducing the problem to $(n - 3)$ discs that remain to be sorted. This process requires $2^{n-2} + 2^{n-3}$ moves.

The next time through the algorithm, we reduce the stack of $(n - 3)$ to $(n - 6)$ discs, requiring $2^{(n-3)-2} + 2^{(n-3)-3} = 2^{n-5} + 2^{n-6}$ moves. This explains why the 2^{n-4} term is absent. Subsequently, we reduce a tower of $(n - 6)$ alternately colored discs by moving stacks of $(n - 8)$ and $(n - 9)$, thus skipping the 2^{n-7} term.

Each time through the algorithm reduces the number of discs in the stack by three, but the number of moves in the final stage will depend on whether or not the original number n was a divisible by 3. When n is a multiple of 3, we will ultimately be left with a stack of three discs to sort, but the largest of the three will already be in position, so it will take 3 moves ($2^1 + 2^0$) to finish. When n is not a multiple of 3 (as with $n = 5$), we may be left with two discs that remain to be sorted, but the larger of the two discs will already be in position, so only 1 more move is needed.

Following this procedure, we can construct **table 1**.

Notice that the exponents in the terms not represented in the sum are bounded by $0 \leq n - (3k + 1) \leq n - 4$, so $(n - 1) \geq k \geq 1$. Because k is an integer, the upper bound is the greatest integer $\left\lfloor \frac{n-1}{3} \right\rfloor$.

If we wish to determine the number of moves for n discs we have

$$\sum_{k=0}^{n-2} 2^k - \sum_{k=1}^{\left\lfloor \frac{n-1}{3} \right\rfloor} 2^{n-1-3k},$$

or

$$(2^{n-1} - 1) - \sum_{k=1}^{\left\lfloor \frac{n-1}{3} \right\rfloor} 2^{n-1-3k}.$$

For example, for the tower of 7 discs, the number of moves required is

$$(2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0) - (2^3 + 2^0) = \sum_{k=0}^5 2^k - \sum_{k=1}^2 2^{6-3k}.$$

This simplifies to $(2^6 - 1) - (2^3 + 2^0) = 54$.

A different way to determine the number of moves uses binary notation. **Table 2** shows the number of moves required for a given tower size, with the number of moves written in binary.

In general, to write the number of moves for a tower of n discs, write 110 repeatedly, stopping when $n - 1$ digits have been written. We can see why this procedure works by reviewing the

algorithm. When we started with n discs, we first moved $n - 2$ discs, using $2^{n-2} - 1$ moves. Moving the single disc to a side post brought the total moves to 2^{n-2} . We next needed to move $n - 3$ discs to a side post, using $2^{n-3} - 1$ moves, which also gave us another proper disc on the side post. We then moved a single disc for 2^{n-3} moves. This initial round involving a total of $2^{n-2} + 2^{n-3}$ moves added three correct discs to the one disc that was in the correct position from the start. Note that since the algorithm will next require a move of $n - 5$ discs, the term 2^{n-4} is absent from the sum. The number of moves so far is $1 \cdot 2^{n-2} + 1 \cdot 2^{n-3} + 0 \cdot 2^{n-4}$, giving rise to the binary digits 110. Since the process continues as the algorithm enters each new round, we see the repeating pattern 110. The solution for 7 discs is $110110_2 = 54$ moves.

If we impose the condition that the discs must be sorted by color onto the outside posts, then the number of moves required for a tower of 7 discs is 1011011_2 , or 91 moves. The general solution to this problem for a tower of n discs that must be moved from the center post to the outside posts is to write 101 repeatedly, stopping when n digits have been written. The derivation of this result is left as an exercise for readers.

Returning to the original Tower of Hanoi challenge, if we record the number of moves in binary notation, then a tower of n discs requires a binary number consisting entirely of n ones. A tower of 7 discs requires 1111111_2 moves, which translates to 127 moves. Since the number following 1111111_2 is 10000000_2 , which is 2^7 , the number of moves can be expressed as $2^7 - 1$.

A THREE-COLOR CHALLENGE

Consider yet another twist on the Tower of Hanoi and use nine discs with three alternating colors, as shown in **figure 2**. The challenge is to rearrange the discs so that each post has discs of the same color.

Written in binary notation, the solution for a tower of n discs is a repeating string of 10 until $n - 1$ digits have been written. We can see why this procedure works by reviewing the process. When we start with n discs, we first move $n - 2$ discs, using $2^{n-2} - 1$ moves, so that we could then use a single move to put a disc in proper position. We next move $n - 4$ discs, so we again leave behind a single disc that can then be moved to the proper place. The algorithm involves moving $n - 2$ discs, then a single disc, then $n - 4$ discs, then a single disc, then $n - 6$ discs, then a single disc, and so on, so that the number of moves is a sum of powers of 2 whose exponents differ by 2, giving rise to the binary 10.

For another challenge, start with the nine discs of three colors, as we just did, but place all the

Table 2 Results Expressed in Binary Notation	
Number of Discs in Original Tower	Number of Moves (Base 2) to Separate Colors
1	0
2	1
3	11
4	110
5	1101
6	11011
7	110110
8	1101101
9	11011011
10	110110110
11	1101101101
n	110 110 110 110 110 ... ($n - 1$ digits)

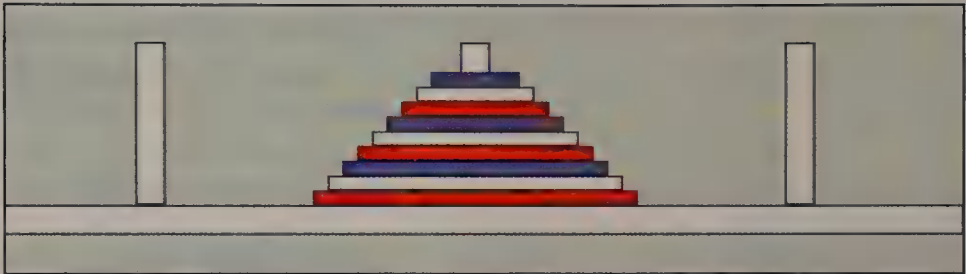


Fig. 2 A nine-disc tower of three colors can be separated in $10101010_2 = 170$ steps.

red discs on one post and the alternating blue and white discs on another post.

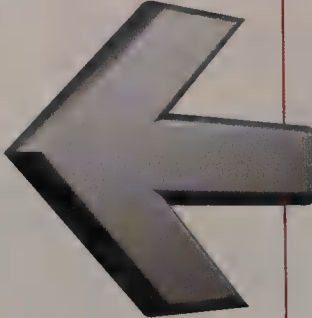
More patterns can be found by using additional colors and pegs. Enjoy the journey.



JAMES METZ, metz@hawaii.edu, a retired mathematics instructor, has conducted mathematics workshops in South Africa with Teachers across Borders and has done similar work in Uganda; his most recent volunteer work was in Liberia with the Peace Corps.

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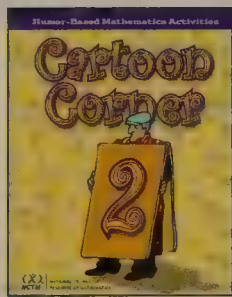


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Cartoon Corner 2, Peggy A. House, ed., 2013. Grades 6-12, 120 pp., \$28.95 paper. ISBN 978-0-87353-718-6. Stock no. 14373. National Council of Teachers of Mathematics; www.nctm.org.



Cartoons serve as the basis for discussion of many and varied mathematics concepts. Each cartoon is accompanied by a set of questions for delving further

into the topic. The author includes the solutions for the worksheets along with responses from teachers who have used them in their own classrooms. She also suggests ideas for extensions.

For the most part, the activities are well designed. Some of the cartoons are a bit obscure but still understandable. Not all the activities would be useful in all 6-12 classrooms, but a range of activities is provided for each level. The activities could be used even in lower-level mathematics classes as long as support is provided.

The book is well written and well

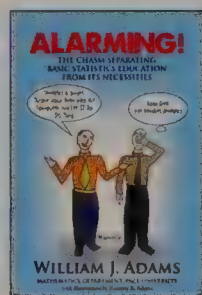
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planned. The activities could provide an alternative activity for the mathematics topics covered in the classroom.

—Jennifer Gonnering
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FROM OTHER PUBLISHERS

Alarming! The Chasm Separating Basic Statistics Education from Its Necessities, William J. Adams, 2013. 193 pp., \$19.99 paper. ISBN 978-1-4771-2011-8. Xlibris; www.xlibris.com.



This book makes two primary and critical points about the teaching of statistics. First, statistical tests make assumptions that are too often ignored or at least not emphasized suffi-

ciently. Second, people should be cautious about taking data at face value. How were the data obtained? What type of sampling was used, and what was the response rate?

Adams is knowledgeable about statistics, and for the most part the book is accurate. However, there are some problems. In many places, he says that when the p -value is above alpha, the user can accept the null hypothesis. Instead, the conclusion should be something to the effect that insufficient evidence was found to reject the null hypothesis and that no conclusions can be drawn. There is no discussion of statistical power or even the fact that a larger N may have yielded statistical significance. The argument concerning the average of ordinal variables (on p. 132) is incorrect. When the standard deviation of the population is unknown, we should use the

t -distribution, not z . In addition, there are numerous typographic errors throughout the book.

The author misses the boat about the use of technology. Requiring manual calculation at all times can and often will make the course quite tedious and restricted in its applications. He often includes formulas without providing their significance. For example, he calculates z -scores and t -values without discussing their derivations and components. Technology can be used not only to facilitate the computation but also to help students visualize, explore, and comprehend many of the ideas that the author laments students are not learning from courses that use technology.

Much discussion in the book is focused on the use of old data sets, some of which are decades old, even though Adams says that more recent data are available. Further, his main point is that different people draw different conclusions from the same data, but he does not elaborate on how the conclusions are drawn or what the outcomes of these conclusions might be or have been. He tends to discuss problems without offering suggestions for improvement, as with data concerning the Consumer Price Index and broader discussions of economics in general. Finally, the book would be stronger with more emphasis on confidence intervals rather than hypothesis testing.

This book might be valuable for those preparing to teach statistics courses, particularly those who do not have advanced degrees in the subject but have taken only a few courses. Students just learning statistics would not find the book particularly helpful.

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MY FAVORITE
lesson

Victoria Miles

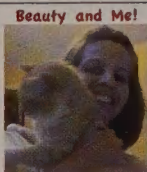
Algebra about Me

Performance tasks provide effective ways to differentiate mathematics instruction while allowing students to be creative and to incorporate mathematics content that is suitable for their interest and readiness. A project that I have enjoyed assigning to my high school students, Algebra about Me, is designed to introduce, reinforce, and review equation-solving concepts and skills (for a customizable activity sheet, go to www.nctm.org/mt).

I first show the class an example that I created about myself (see **fig. 1**), a fun way to share some information about me. If students want to know how many hours, h , that my cat typically sleeps in a day, they must correctly solve an equation: $h + 2 = 3^2 - 4^2 + 5^2$. To discover my house number, n , students solve $n^4 = 81$. Students can determine the number of letters in my first name, f , by solving $-2f + 10 = -6$.

These basic equations provide a high-interest way to review critical content with students who have already completed first-year algebra and allow me to introduce equation-solving concepts and skills in an engaging way with current first-year algebra students.

Students work in cooperative groups to solve the equations in my sample project. Solving the equations cooperatively addresses the Common Core's Standards for Mathematical Practice, particularly reasoning abstractly and quantitatively, constructing viable arguments, and cri-

Algebra About Me!	
There are "k" keys on my piano. $(k + 4) + 3 = 10^2 \div 4$ 	My cat Beauty sleeps h hours each day. $h + 2 = 3^2 - 4^2 + 5^2$ 
Beauty and Me! 	I live on "n" Tinkers Lane $n^4 = 81$ 
At my house, the sum of the ages is "m". $100 - m = 3$ 	I have "f" letters in my first name. $-2f + 10 = -6$ 

by: Ms. Miles

Fig. 1 A personalized project creates interest.

tiquing the reasoning of others. I use a document camera to project student solutions to my equations so that classmates can verify or question one another's work.

Next, we write a variety of equations in class. If the average age of the students in a class is 16, we brainstorm various equations whose solutions would be 16. I might write on the board, "I am y years old." Here are examples of student-generated equations that result from performing one or more operations on the equation $y = 16$:

$$\text{Solve: } y + 4 = 20$$

$$\text{Solve: } y + 4 = 40 \div 2$$

$$\text{Solve: } \sqrt{y} = 4$$

$$\text{Solve: } 2y - 7 \cdot 5 = -3$$

Student-generated problems can involve an expression whose value is 16, such as this: Evaluate $a - 2b + 10$ if $a = -4$ and $b = -5$.

For homework that night, students complete a rough draft of their own Algebra about Me project. The next day in class, they exchange drafts with

a partner. Peers help one another edit their equations for mathematical correctness as well as their written work for grammar and structure.

I typically assign this project early in the year and display the projects on my interactive bulletin board so that students can solve equations in their spare moments in class while getting to know fun facts about their classmates. The project directions provide scoring details, so students are able to self-assess their work. This informative project gives me a clear picture of each student's equation-solving background and algebraic reasoning skills.

Some students choose to create final projects digitally; others use creative physical materials and incorporate artwork. Students who enjoy being challenged often write equations and problem statements that are truly impressive. These projects make a colorful bulletin board that can be showcased during a school open house or parent conferences.



VICTORIA MILES, victoria.miles1@gmail.com, teaches algebra and geometry at Middleboro High School in Middleboro, Massachusetts. She is interested in problem solving and geometry and enjoys providing and participating in professional development workshops.

The Back Page provides a forum for readers to share a favorite lesson. Lessons to be considered for publication should be submitted to mt.msubmit.net. Lessons should not exceed 600 words and are subject to abridgment.

Edited by **Roger Day**, day@ilstu.edu,
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more **4U**

For images of students' artwork, download one of the free apps for your smartphone and then scan this tag to access www.nctm.org/mt055. For an activity sheet available to teachers as a Word document that can be copied and edited for classroom use, go to www.nctm.org/mt.



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